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MATRIX METHOD FOR OBTAINING SPANWISE MOMENTS AND DEFLECTIONS OF TORSIONALLY RIGID ROTOR BLADES WITH ARBITRARY LOADINGS

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SUMMARY

A matrix solution for the spanwise bending moments and deflections of a torsionally rigid rotor blade subjected to an arbitrary loading is presented. The method includes the cantilever, teetering, and hinged blades in hovering and in steady forward flight. The method is comparatively short, involves only standard matrix procedures, and does not require that the mode shapes or natural frequencies be known.

INTRODUCTION

Numerous methods are available for calculating the uncoupled spanwise bending moments and deflections of rotor blades in steady unaccelerated flight. Some of these methods (refs. 1 and 2) involve tabular solutions of the differential equation of blade deformation. These tabular procedures are very lengthy, particularly when numerous loading conditions and rotor speeds are being investigated. Other methods (refs. 3 and 4) require the calculation of the natural-mode shapes and frequencies of the blade. The calculations of mode shape and frequency are laborious and either must be repeated for each rotor speed or an approximate correction must be applied to account for the effects of rotor speed. A matrix method which avoids some of the difficulties of the tabular and modal solutions is presented in reference 5; however, tabular solutions of the equation for blade bending moment under quasi-static conditions are required before the method can be applied.

In the present method the differential equation for blade bending moment is solved entirely by matrix procedures. The method permits the determination of the bending moments and deflections directly without preliminary quasi-static, mode-shape, or natural-frequency calculations and, as a result, the method is shorter and much work is eliminated. The method is adaptable to cantilever, semirigid (teetering), and fully articulated (hinged) blades in hovering and in steady forward flight and is applicable to design calculations and experimental data analyses.

The differential equation for the blade bending moment is composed of matrix expressions for the centrifugal inertia, vertical inertia, and damping loadings, and the contribution of each term to the blade loads and moments may be determined. The effect of blade structural damping, which has been omitted in other methods, is included in the present method. Comparisons are presented between the bending moments obtained by this method and the method of reference 3 for the first harmonic loading on a fully articulated blade and the second harmonic loading on a cantilever blade.

SYMBOLS

Some of the symbols listed in this section are illustrated in figure 1.

A blade-element average aerodynamic damping coefficient, $-\frac{1}{2}\rho ac\,\Delta r\,r$, $\frac{\text{lb-sec}^2}{\text{in.}}$

a blade-element lift-curve slope, per radian

c blade chord (at center of blade element), in.

EI blade bending stiffness, lb-in.^{2}

F centrifugal force acting on mass of blade element divided by Ω^2, lb-sec^2

g structural damping coefficient

h perpendicular distance from axis of rotation to flapping hinge, in.

i imaginary component of complex number, $\sqrt{-1}$

l blade running load, lb/in.

L load on blade element, lb

M bending moment, lb-in.

m	mass of blade element, $\frac{\text{lb-sec}^2}{\text{in.}}$
n_e	number of blade elements for extended matrices
R	distance from center of hub to blade tip, in.
r	spanwise distance from center of hub (measured along undeflected blade), in.
Δr	width of blade element, in.
V	forward velocity, in./sec
Z_M	elements of an influence coefficient matrix (see table III)
z	blade deflection, measured from a plane of rotation perpendicular to shaft axis, in.
β	rigid-blade flapping angle, measured from a plane of rotation perpendicular to shaft axis, radians
β_h	angle of flexible blade at hinge, measured from a plane of rotation perpendicular to shaft axis, radians
β_r	blade slope due to blade bending, dz/dr, radians
Ω	rotor angular velocity, radians/sec
ω	angular frequency of applied loading, radians/sec
ρ	air density, $\frac{\text{lb-sec}^2}{\text{in.}^4}$
ψ	blade azimuth angle (measured in direction of blade rotation from downwind position), radians

Subscripts:

a	aerodynamic
C	cosine component
d	damping
H	due to forces parallel to plane of rotation
i	inertia

L	pertaining to a load station
r	pertaining to a spanwise station
s	structural
S	sine component
1,2,...n	station number or number of harmonic
V	due to forces perpendicular to the plane of rotation
w	blade weight
z	deflection
f	centrifugal loading

A dot over a symbol indicates first derivative with respect to time. Two dots indicate second derivative with respect to time.

Matrix notation:

$\left[\ \right]$	rectangular or square matrix
$\lceil\ \rfloor$	diagonal matrix
$\left\{\ \right\}$	column matrix
$\left[\ \right]^{-1}$	inverse of a square matrix
$\lfloor\ \rfloor$	row matrix

METHOD

The differential equation for the structural bending moments of hinged, teetering, and cantilever rotor blades is set up in matrix form. Matrix expressions for the blade moment due to the various blade loadings are derived and added to form the blade-moment equation. The differential equation is first derived in a generalized form and is later adapted to (1) the case where the rigid-blade aerodynamic loads are known from design calculations and the flexible-blade structural moments or deflections, or both, are desired; (2) the case where the blade structural moments are known from strain-gage tests and the total aerodynamic moment, including aerodynamic damping, is desired; and (3) the case where the aerodynamic loads have been measured and the structural moments are desired. The procedure for obtaining the rotating-blade characteristic matrix is also presented so that the natural modes and frequencies may be determined.

Blade Loadings

In the calculation of spanwise bending moments, a torsionally rigid rotor blade or its equivalent (an elastic blade for which the section moments are zero and the aerodynamic center, elastic axis, and center of gravity coincide) may be treated as a simple beam subject to numerous superimposed loadings. If chordwise effects are excluded, the blade in the forward flight condition is subjected to the following distributed loads:

(1) The aerodynamic load L_a (excluding the aerodynamic damping effect which is directly proportional to blade flexure plus flapping deflection velocity; see item 5)

(2) The centrifugal load F

(3) The blade weight L_w

(4) A normal inertia loading due to vertical accelerations L_{iV}

(5) An aerodynamic damping load L_{da} (defined by eq. (5)), directly proportional to blade deflection velocity due to both flapping and flexure

(6) The blade structural damping load L_{ds}

The combined moments of these six loadings are balanced by the blade internal moment due to blade bending resistance.

Derivation of Blade Moment Equations

The blade geometry, the forces acting on an infinitesimal element of the blade under forward-flight conditions, and the resulting force system are shown in figure 1. From figure 1 it is seen that the bending moment M_{r_n} at any station r_n due to the loads acting on the blade is given by:

$$M_{rn} = \int_{r_n}^{R} l_V(r - r_n)dr - \int_{r_n}^{R} l_H(z - z_n)dr \tag{1}$$

The matrix equivalent of equation (1) may be written in the form

$$\{M\} = [r]\{L_V\} - \Omega^2[F]\{z\} \tag{2}$$

Equation (2) corresponds to a step-diagram representation of the l_V and l_H loadings of equation (1). The $[r]$ matrix is an integrating matrix with values that depend on the stations and integration method used. The $[F]$ matrix incorporates the values of the horizontal loads $\{L_{iH}\}$ (dimensional with respect to rotor speed) as well as integrating numbers. The $[r]$ and $[F]$ matrices are derived in the appendix.

Substituting the five components of $\{L_V\}$ enumerated previously into equation (2) yields the following bending-moment equation of the combined loadings:

$$\{M_r\} = [r]\{L_a\} + [r]\{L_w\} + [r]\{L_{da}\} + [r]\{L_{iV}\} + [r]\{L_{ds}\} - \Omega^2[F]\{z\} \tag{3}$$

Equation (3) is the general form of the equation for blade bending moment used in this study and pertains to hinged and cantilever blades in hovering and forward flight. In order to obtain the blade bending moment for any specific loading condition, the matrix expressions for the various loads acting are substituted into equation (3).

Loading matrices.- The expressions for the various component loads in equation (3) become quite simple if equation (3) is restricted to the expression of the moments due to loadings of a single frequency. The total aerodynamic loads on the blade may be broken down into components which are of a sinusoidal nature, and the standard procedure is to develop equation (3) for the moment due to a single frequency. Superposition may then be used to obtain the total moments due to all of the harmonic components on the blade. The loading matrices which appear in equation (3) are given by the three following equations.

For a particular frequency ω of the applied sinusoidal load, the vertical inertia loads $\{L_{iV}\}$ are given by

$$\left\{L_{iV}\right\} = \omega^2 \left[m\right] \left\{z\right\} \tag{4}$$

where the elements of the $\left[m\right]$ matrix are the masses of the blade elements.

The aerodynamic damping loads $\left\{L_{da}\right\}$ are given by

$$\left\{L_{da}\right\} = i\Omega\omega \left[A\right] \left\{z\right\} \tag{5}$$

where the diagonal matrix $\left[A\right]$ is equal to $\left[-\frac{1}{2} \rho acr \Delta r\right]$

The elements of the $\left[A\right]$ matrix can be deduced from steady aerodynamics for the lift on a section of chord c and length Δr rotating at unit angular velocity and with a unit normal velocity.

The structural damping moment $\left[r\right]\left\{L_{ds}\right\}$ is given by

$$\left[r\right]\left\{L_{ds}\right\} = -ig \left[1.0\right] \left\{M_s\right\} \tag{6}$$

Moments and deflections at a single frequency.- The blade loadings at a particular frequency ω (eqs. (4), (5), and (6)) may be substituted into equation (3) in order to obtain the equation for the blade moment in terms of the structural moment, the aerodynamic moment, and the blade deflections as

$$\left[1 + ig\right] \left\{M_s\right\} = \left[r\right]\left\{L_a\right\} + i\Omega\omega \left[r\right] \left[A\right] \left\{z\right\} + \omega^2\left[r\right] \left[m\right] \left\{z\right\} - \Omega^2\left[F\right] \left\{z\right\} \tag{7}$$

The deflection $\left\{z\right\}$ in equation (7) is composed of the flexural bending and the effects of blade-root rotation. The flexural bending is defined in the appendix as

$$\left\{z\right\} = \left[Z_M\right]\left\{M_s\right\} \tag{8}$$

The total blade deflection including root rotation is, therefore, given by

$$\left\{z\right\} = \left[Z_M\right] \left\{M_s\right\} + \left\{r - h\right\} \tan \beta_h \tag{9}$$

For small angles $\tan \beta_h \approx \beta_h$, and in the equations to follow, which are based on assumptions of small angles, β_h is assumed equal to $\tan \beta_h$. Equation (9), therefore, becomes

$$\{z\} = \left[\left[Z_M\right] \vdots \{r - h\}\right] \left\{\begin{array}{c}\{M_s\} \\ \hdashline \beta_h\end{array}\right\} \tag{10}$$

Substituting equation (10) into equation (7) yields the following relation between the aerodynamic loads and the structural moments for the flexible blade:

$$\left[(1.0 + ig)\left[\begin{array}{c:c}\left[1.0\right] & \{0\} \\ \hdashline \left[0\right] & 0\end{array}\right] + \Omega^2[F]\left[\left[Z_M\right] \vdots \{r - h\}\right] - \omega^2\left[M_{zV}\right]\left[\left[Z_M\right] \vdots \{r - h\}\right] - \right.$$

$$\left. i\omega\Omega[r][A]\left[\left[Z_M\right] \vdots \{r - h\}\right]\right]\left\{\begin{array}{c}\{M_s\} \\ \hdashline \beta_h\end{array}\right\} = [r]\{L_a\} \tag{11}$$

where

$$\left[M_{zV}\right] = [r][m]$$

The partition lines of equation (11) effectively separate structural moment effects from root slope effects, both of which act to balance the aerodynamic moment. If the root slope β_h is set equal to zero, the terms pertaining to the root slope drop out and the resulting equation is the equation for a cantilever-blade bending moment. The cantilever-blade equations are given in detail in a subsequent section.

Equation (11) may be inverted in order to obtain the reciprocal matrix relation

$$\left\{\begin{array}{c}\{M_s\} \\ \hdashline \beta_h\end{array}\right\} = \left[(1.0 + ig)\left[\begin{array}{c:c}\left[1.0\right] & \{0\} \\ \hdashline \left[0\right] & 0\end{array}\right] + \Omega^2[F]\left[\left[Z_M\right] \vdots \{r - h\}\right] - \right.$$

$$\left. \omega^2\left[M_{zV}\right]\left[\left[Z_M\right] \vdots \{r - h\}\right] - i\omega\Omega[r][A]\left[\left[Z_M\right] \vdots \{r - h\}\right]\right]^{-1}\{M_a\} \tag{12}$$

where

$$\{M_a\} = [r]\{L_a\}$$

Equation (12) gives the structural moment $\{M_s\}$ and root slope β_h on the rotating hinged blade in terms of the applied aerodynamic moment $\{M_a\}$. The equation includes the effect of the aerodynamic damping, which is proportional to the total deflection velocity.

When calculations are to be made for many rotor speeds, it is advantageous to put equation (12) in the form

$$\begin{Bmatrix} \{M_s\} \\ \hline \beta_h \end{Bmatrix} = \left[(1.0 + ig)\left[\begin{array}{c|c} [1.0] & \{0\} \\ \hline \lfloor 0 \rfloor & 0 \end{array}\right] + \Omega^2[N]\right]^{-1}\{M_a\} \tag{13}$$

where the matrix $[N]$ is evaluated for each harmonic, and

$$[N] = \Bigg[[F]\Big[[Z_M]\Big|\{r - h\}\Big] - \frac{\omega^2}{\Omega^2}[M_{zV}]\Big[[Z_M]\Big|\{r - h\}\Big] - i\,\frac{\omega}{\Omega}[r][A]\Big[[Z_M]\Big|\{r - h\}\Big]\Bigg] \tag{14}$$

The matrix $[N]$ need be evaluated only for each harmonic number.

Hovering condition.- Equation (12), which was derived for the forward-flight condition, would also apply to the hovering condition if the terms which pertain to the blade vertical velocity and acceleration are omitted. Because the hovering aerodynamic loads are steady, $g = 0$ and $\omega = 0$; and the equation for hovering becomes

$$\begin{Bmatrix} \{M_s\} \\ \hline \beta_h \end{Bmatrix} = \left[\left[\begin{array}{c|c} [1.0] & \{0\} \\ \hline \lfloor 0 \rfloor & 0 \end{array}\right] + \Omega^2[F]\Big[[Z_M]\Big|\{r - h\}\Big]\right]^{-1}\{M_a\} \tag{15}$$

Cantilever-blade equations.- If the blade-root angle β_h is zero in equations (12) and (15), the result is the equation for the cantilever blade in forward flight:

$$\left\{M_a\right\} = \left[\left[1.0\right] + ig\left[1.0\right] + \Omega^2\left[F\right]\left[Z_M\right] - \omega^2\left[M_{zV}\right]\left[Z_M\right] - i\omega\Omega\left[r\right]\left[A\right]\left[Z_M\right]\right]\left\{M_s\right\} \tag{16}$$

or

$$\left\{M_s\right\} = \left[\left[1.0\right] + ig\left[1.0\right] + \Omega^2\left[F\right]\left[Z_M\right] - \omega^2\left[M_{zV}\right]\left[Z_M\right] - i\omega\Omega\left[r\right]\left[A\right]\left[Z_M\right]\right]^{-1}\left\{M_a\right\} \tag{17}$$

If the calculations are to be made for numerous rotor speeds, equation (17) may be put in the same form as equation (13).

The cantilever-blade equation for the hovering condition is:

$$\left\{M_s\right\} = \left[\left[1.0\right] + \Omega^2\left[F\right]\left[Z_M\right]\right]^{-1}\left\{M_a\right\} \tag{18}$$

CALCULATION PROCEDURES

Matrix Evaluation

The calculation procedures presented are illustrated for the hinged blade. The details given pertain to the ten-segment breakdown of the blade load, weight, and stiffness distribution shown in figure 2. The procedure for either a cantilever or a teetering blade is a simplification of the hinged-blade case. The particular blade-section breakdown in the matrix tables (see figs. 2 and 3) was selected to provide more stations near the root of the blade where rapid changes in blade moment and stiffness occur. The number of blade elements used may be increased by the procedures given in the appendix for increasing the order of each matrix. However, it is believed that for most blades ten stations are sufficient.

In order to apply the hinged-blade equations, it is first necessary to evaluate the various matrices in the equations. The hinged-blade equation, previously given as equation (12), is

$$\begin{Bmatrix} \{M_s\} \\ \hline \beta_h \end{Bmatrix} = \left[(1.0 + ig) \left[\begin{array}{c|c} [1.0] & \{0\} \\ \hline \lfloor 0 \rfloor & 0 \end{array} \right] + \Omega^2 [F] \left[[Z_M] \,\middle|\, \{r - h\} \right] - \right.$$

$$\left. \omega^2 [M_{zV}] \left[[Z_M] \,\middle|\, \{r - h\} \right] - i\omega\Omega [r] [A] \left[[Z_M] \,\middle|\, \{r - h\} \right] \right]^{-1} \{M_a\}$$

The structural damping coefficient g is a function of the frequency, and the value used should correspond to the value of ω for which the equation is being evaluated. The matrix $\left[\begin{array}{c|c} [1.0] & \{0\} \\ \hline \lfloor 0 \rfloor & 0 \end{array} \right]$ is an identity matrix where the zero, as the last element on the diagonal, imposes the condition of zero structural moment at the blade hinge. This matrix is given in table I. The matrix $[F]$ expresses the moment on the blade caused by the centrifugal forces acting on the deflected blade. The $[F]$ matrix is given in table II. The matrix $\left[[Z_M] \,\middle|\, \{r - h\} \right]$ expresses the blade deflection caused by the structural moment and the root slope. This matrix is given in table III. The matrix $[M_{zV}]$ expresses the vertical inertia moment in terms of the blade deflection. The $[M_{zV}]$ matrix is given in table IV. The $[r]$ matrix is an integrating matrix which expresses the moment of a loading and is given in table V. The matrix $[A]$ expresses the aerodynamic damping load in terms of the blade deflection. The matrix $[A]$ is given in table VI.

The number of digits needed in the evaluation of the various matrices and in the succeeding operations for the most accurate results has not been rigorously established. However, limited experience has shown that carrying the basic masses, distances, stiffnesses, and so forth to four digits and, in succeeding operations, allowing the number of digits to accumulate to a maximum of ten gives good results. The procedure for filling out the given matrix tables is as follows:

(1) Set up the identity matrix as shown in table I, which is already completed in its entirety and may be used as is.

(2) Obtain the weight distribution for the blade and break it down into ten elements of equal length. Determine the mass of each element. Use the product mr for each element as described in the appendix, and fill out table II.

(3) Obtain the stiffness distribution for the blade and break it down into the smaller elements shown in figure 2. Use these stiffness values and the values of $\overline{\Delta r^2}$ and fill out table III.

(4) Use the mass values of item 2 and fill out table IV.

(5) Substitute the value of Δr into table V.

(6) Use the values of the chord of each element, the lift-curve slope of each element, the radius to the center of each element, and the values of ρ and Δr, and fill out table VI.

Application to the Hinged Blade

The equations for the hinged blade are applied to those problems which are believed to be the most general. The cases discussed include the determination of (1) the flexible-blade structural moment due to the rigid-blade aerodynamic loads; (2) the structural moment due to the measured aerodynamic loads; (3) the moment of the total aerodynamic loading as determined from the measured structural moment; and (4) the aerodynamic damping load. The procedure for obtaining the natural modes and frequencies of the blade is also discussed since, in some cases, this information may be desired.

Structural moment due to rigid-blade aerodynamic loads.- The standard design practice is to obtain the rigid-blade total loading from rigid-blade aerodynamic load and inertia balance equations. The rigid-blade aerodynamic and inertia loads are then applied to the flexible blade which is considered to flex about the rigid-blade position. This procedure is equivalent to the application of the rigid-blade aerodynamic load only (minus an aerodynamic damping component which is proportional to $\dot{z}$) to the equation for the flexible blade which was derived as equation (12) and in which the deflections are referenced to the plane of rotation perpendicular to the shaft. The equivalence of the standard design procedure and the procedures of the present method is proved by the following:

The rigid-blade total load (aerodynamic plus inertia) which is normally applied to the flexible blade (considered to flex about the rigid-blade position) is given by

$$L_{\substack{rigid\\total}} = \left(L_a\right)_{\substack{rigid\\total}} - mr\ddot{\beta} - m\omega^2 r\beta \qquad (19)$$

where the terms on the right-hand side of the equation represent the rigid-blade aerodynamic, vertical-inertia, and centrifugal-inertia loads, respectively. Since the equations of the present method are written in terms of moment, it is useful to convert equation (19) to its moment equivalent. Also, it is useful to express each term in equation (19) in the matrix form. Thus, equation (19) becomes

$$\left\{\begin{matrix} M_{rigid} \\ total \end{matrix}\right\} = [r]\left\{\begin{matrix}(L_a)_{rigid} \\ total\end{matrix}\right\} + \omega^2[M_{zV}]\{z_{rigid}\} - \Omega^2[F]\{z_{rigid}\} \tag{20}$$

where the terms on the right-hand side of the equation are the rigid-blade aerodynamic, vertical-inertia, and centrifugal-inertia moments, respectively.

If the rigid-blade moments given by equation (20) are applied to the flexible blade, which is considered to flex about the rigid-blade position, the following equation results:

$$\overbrace{[r]\left\{\begin{matrix}(L_a)_{rigid} \\ total\end{matrix}\right\} + \omega^2[M_{zV}]\{z_{rigid}\} - \Omega^2[F]\{z_{rigid}\}}^{\text{Moment of rigid-blade loading}} =$$

$$\overbrace{(1.0 + ig)\{M_s\} + \Omega^2[F]\{z_{flex}\} - \omega^2[M_{zV}]\{z_{flex}\} - i\Omega\omega[r][A]\{z_{flex}\}}^{\text{Moment of flexible-blade loading}} \tag{21}$$

The terms on the right-hand side of the equation give the moments associated with the blade deflection $\{z_{flex}\}$ about the rigid-blade position. These terms, from left to right, represent the blade-bending, the centrifugal-inertia, the vertical-inertia, and the aerodynamic-damping moment, respectively. Equation (21) implies that the blade moments resulting from the flexible-blade deflections must balance the applied rigid-blade moment. The flexible- and rigid-blade deflections z_{flex} and z_{rigid} are shown in figure 1(b). If it is recognized that z_{flex} plus z_{rigid} is equal to the deflection z (measured with respect to the plane of rotation) and if the terms of equation (21) are collected, then the following equation results:

$$[r]\left\{\begin{matrix}(L_a)_{rigid} \\ total\end{matrix}\right\} = (1.0 + ig)\{M_s\} + \Omega^2[F]\{z\} - \omega^2[M_{zV}]\{z\} - i\Omega\omega[r][A]\{z_{flex}\} \tag{22}$$

Adding the term $-i\Omega\omega[r][A]\{z_{rigid}\}$ to both sides of equation (22), collecting terms, and rearranging yields

$$(1.0 + ig)\{M_s\} = [r]\{L_a\} + i\Omega\omega[r][A]\{z\} + \omega^2[M_{zV}]\{z\} - \Omega^2[F]\{z\} \tag{23}$$

where

$$\{L_a\} = \left\{\begin{matrix}(L_a)_{rigid} \\ \text{total}\end{matrix}\right\} - i\Omega\omega[A]\{z_{rigid}\} \tag{24}$$

Equation (23) is recognizable as equation (7) where $\{L_a\}$ is defined as the total rigid-blade aerodynamic load minus a component which is proportional to the rigid-blade deflection velocity. The matrix $[M_{zV}]$ is equal to the matrix multiplication $[r][m]$ shown in equation (7).

Substituting $\{z_{rigid}\} = \{r - h\}\beta$ in equation (24) yields

$$\{L_a\} = \left\{\begin{matrix}(L_a)_{rigid} \\ \text{total}\end{matrix}\right\} - i\Omega\omega[A]\{r - h\}\beta \tag{25}$$

The flexible-blade moments, therefore, may be determined by applying equation (25) (the rigid-blade total aerodynamic load minus a rigid-blade deflection-velocity load) to the flexible-blade equation (eq. (12)) which is referenced to the plane of rotation.

In order to obtain the flexible-blade structural moment due to rigid-blade airloads:

(1) Obtain the rigid-blade total aerodynamic load for each harmonic and remove the flapping effects by using equation (25).

(2) Use the loading just obtained along with the previously determined matrices, and obtain the moment by means of equation (12).

The reason the rigid-blade loads were handled in this manner was to keep the blade equations referenced to the plane of rotation in order to provide for more general applications.

<u>Structural moment due to measured airloads</u>.- The procedure for the case of measured airloads is as follows: Since measured loads include all aerodynamic damping effects, the aerodynamic damping term is dropped from equation (12) and the aerodynamic moment $\{M_a\}$ is given by

$\{M_a\} = [r]\{L_{measured}\}$. Equation (12) is then applied by use of the predetermined matrices to obtain the structural moment.

If the hinged-blade equations are used with measured airloads where $\omega = \Omega$ and the equations involve no aerodynamic damping, the last row and last column automatically drop out of equations (11) and (12) when the matrix multiplications are performed, and β_h is no longer involved in the solution. This behavior of the equations is consistent with the physical concept of blade rotation in the plane of no flapping or rigid mode resonance.

Moment of total aerodynamic loading as determined from measured structural moment.- A requirement for determining the total aerodynamic moment is that the blade flapping angle at the root of the flexible blade β_h be measured as well as the structural moment. The procedure, then, is to apply equation (11) and drop the aerodynamic damping term from the equation because the structural moments are a result of the total loads applied to the blade.

The aerodynamic moment obtained is thus the total aerodynamic moment and includes the moment due to all of the aerodynamic damping loads.

Aerodynamic damping loads.- In some of the foregoing applications it may be desired to determine the aerodynamic damping load or moment which is proportional to the blade total deflection velocity $\dot{z}$ in order to remove these effects from the measured data. The load and moment linearly related to the deflection velocity are given by

$$\left.\begin{aligned} \{L_{da}\} &= i\omega\Omega[A]\left[[Z_M] \;\vdots\; \{r - h\}\right]\begin{Bmatrix}\{M_s\} \\ \hdashline \beta_h\end{Bmatrix} \\ &\text{and} \\ \{M_{da}\} &= i\omega\Omega[r][A]\left[[Z_M] \;\vdots\; \{r - h\}\right]\begin{Bmatrix}\{M_s\} \\ \hdashline \beta_h\end{Bmatrix} \end{aligned}\right\} \tag{26}$$

The values of $\begin{Bmatrix}\{M_s\} \\ \hdashline \beta_h\end{Bmatrix}$ may be either from measurements or from calculations for the lower harmonics, but β_h would probably have to be calculated for the higher harmonics because of its extremely small magnitude.

Determination of blade natural modes and frequencies.- If the blade rotating natural modes and frequencies are desired, the procedure for their determination is as follows:

(1) Solve equation (15) for unit loads applied at the center of each of the ten blade elements.

(2) Use equation (10) and the results of step (1) and determine the deflections due to the unit loads.

(3) Set up the numbers from step (2) in matrix form, postmultiply by the diagonal mass matrix $[m]$, and follow standard iteration procedures. (See ref. 6.)

The ten-element breakdown of the blade allows ten degrees of freedom and thus permits the calculation of ten natural modes and frequencies; however, the use of the standard iteration procedure for obtaining accurate results above the fourth natural frequency is rather difficult because the iterations converge slowly at these higher frequencies.

Application to Cantilever and Teetering Blades

The foregoing procedures were based on the hinged blade. When the problem is for a cantilever blade, the only change is to use the corresponding cantilever-blade equation and matrices. The cantilever-blade equivalents of the hinged-blade equations (10) and (25) are not given but are obtained by making β_h and β equal to zero in these equations. The cantilever-blade matrices are given in tables I through V as the elements above and to the left of the dashed partition lines.

In an analysis of a teetering blade, the even harmonic loadings are considered as applied to a cantilever blade; the odd harmonic loadings are considered as applied to the hinged blade. The reasons for the different considerations for different harmonics is that even harmonic loadings are symmetrical loadings, that is, of equal sign on both sides of the hub; the odd harmonic loadings are unsymmetric.

NUMERICAL EXAMPLES

In order to illustrate the method, it is applied to a cantilever and to a zero-offset hinged blade. Both blades were of the same weight and stiffness, with the exception that the stiffness of the hinged blade was zero at the hinge. The weight and stiffness distributions are given in figure 4.

The aerodynamic loading applied to the hinged blade in forward flight is shown in figure 5(a). The loadings shown are plots against spanwise station of assumed $l_{1,C}$ and $l_{1,S}$ coefficients for the first harmonic of measured blade airloads as defined by

$$l_{1,r} = l_{1,C} \cos \psi + l_{1,S} \sin \psi$$

By using the loading of figure 5(a) and the weight and stiffness of the blade shown in figure 4, the structural moment coefficient shown in figure 5(b) was calculated by means of equation (12). The aerodynamic damping term of equation (12) was dropped because measured loads already included aerodynamic damping effects. The effects of structural damping were not included, although the inclusion of structural damping in the equations should improve the accuracy of the calculations for the case where the blade is near resonance. However, near blade resonance, the effects of small errors in the calculated blade weight and stiffness on the calculated results are likely to be large, and the problem requires extreme accuracy in all components.

The structural moments calculated by the modal method of reference 3 for the loading of figure 5(a) are also shown in figure 5(b). For these calculations the first four natural rotating modes and frequencies were used in equations of reference 3. The aerodynamic damping terms were dropped from the equations. The mode shapes and natural frequencies were obtained by the method previously outlined in the calculation procedures. As can be seen in figure 5(b), the agreement of the results obtained by the two methods is very good.

The aerodynamic loading applied to the cantilever blade was the forward-flight second-harmonic loading shown in figure 6(a). The loadings shown are plots against spanwise station of assumed $l_{2,C}$ and $l_{2,S}$ coefficients for the second harmonic of measured airloads as defined by

$$l_{2,r} = l_{2,C} \cos 2\psi + l_{2,S} \sin 2\psi$$

This loading and the weight and stiffness distribution of figure 4 were used in equation (17) to calculate the structural moments shown in figure 6(b). Also shown in figure 6(b) are the results obtained by the method of reference 3 by using the same aerodynamic load (from fig. 6(a)) and four rotating symmetrical modes and natural frequencies. The results calculated by the present method and by the method of reference 3 are seen in the figure to be in good agreement.

The structural moments on the rotating hinged and cantilever blades caused by 100-pound loads applied at various spanwise stations are given in figures 7 and 8. The moments were obtained by the use of equations (15) and (18). By using the curves for the moment on the cantilever blade caused by 100-pound loads at various stations and the aerodynamic loads measured at various stations in the hovering test of reference 7, the structural moments in hovering were calculated. The calculated moment and the blade deflections for the hovering condition are shown in figure 9.

The various matrices used in the example calculations are shown in tables VII to X. The elements above and to the left of the partition lines pertain to the cantilever blade. The matrix of all of the given elements pertains to the hinged-blade case.

CONCLUDING REMARKS

A matrix method has been derived for determining the structural moment and deflections of hinged, teetering, and cantilever blades. The method avoids any preliminary mode shape, natural frequency, or quasi-static calculations and, as a result, is comparatively short and involves only standard matrix procedures.

The equations are well suited to the determination of the blade bending moment at various rotor speeds and for numerous loadings. The method puts no restriction on the blade deflection shape and thus permits the blade to assume more complex shapes that could be simulated by the superposition of four natural modes, as is usually employed in a modal method of solution of the differential equation for blade bending moment. Thus, the present method should give more accurate results than the modal solution for the higher harmonics, where the blade deflection and load distribution are of a more complex shape.

The method includes the effects of the primary inertia and damping loads on the blade and also incorporates mainly the following assumptions: (1) small angle consideration, (2) step diagram of the blade mass and stiffness, (3) step integration procedures, and (4) steady average aerodynamic considerations in the damping terms. In addition, the method does not include the effects of: (1) radial change of mass position with blade deflection, (2) variation of rotor speed with azimuth angle, and (3) any torsional considerations. Some of the effects of the step distributions may be removed by increasing the number of blade elements; however, for most blades ten elements should be sufficient.

The inclusion of structural damping in the equations should improve the accuracy of the calculations for the case where the blade is near resonance. However, near blade resonance, the effects of small errors in the calculated blade weight and stiffness on the calculated results are likely to be large, and the problem requires extreme accuracy in all components. These unfavorable factors, in some cases, could be somewhat offset by experimental determination of the elements of the influence coefficient matrix $\left[Z_M\right]$ and the mass of the blade elements and by some experimental estimate of the structural damping. In cases where the blade is not near resonance with one of the applied load frequencies, the structural damping term is relatively unimportant and may be dropped from the equations for blade bending moment.

Langley Aeronautical Laboratory,
National Advisory Committee for Aeronautics,
Langley Field, Va., May 7, 1958.

APPENDIX

BASIC INTEGRATING MATRICES

Moment of an Arbitrary Loading

The general bending-moment equation (eq. (1)) is given as

$$M_{rn} = \int_{r_n}^{R} l_V\left(r - r_n\right)dr - \int_{r_n}^{R} l_H\left(z - z_n\right)dr$$

In this section the matrix equivalent of the expression

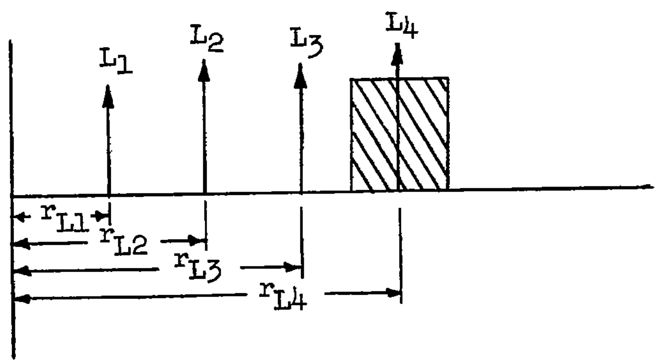
$$M_{rn} = \int_{r_n}^{R} l_V\left(r - r_n\right)dr \tag{A1}$$

is derived.

The integration of equation (A1) may be accomplished by using a procedure based on step, trapezoidal, or parabolic representation of the loading. In the present method the step diagram is used, and the load is considered to act at the center of each step.

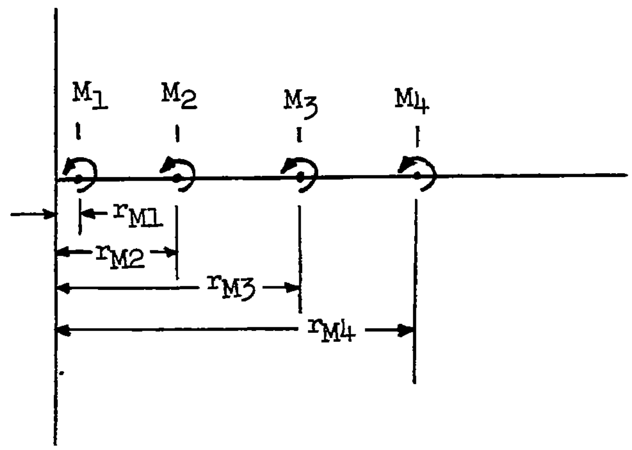
The procedure for obtaining the moment of a loading is as follows:

(1) The loading is replaced by a system of equivalent concentrated loads as shown in the following sketch:



Arbitrary load station arrangement

(2) The moment due to the loading is desired at the stations shown in the following sketch:



Arbitrary moment station arrangement

The loads are centered for 10 equal stations at 0.05(R - h), 0.15(R - h), 0.25(R - h), and so forth, whereas the moments are taken at stations 0.85(R - h), 0.75(R - h), and so forth except near the root where stations at 0.0125(R - h) and 0.0625(R - h) are used to include possible hub effects.

With the arrangement just shown the moment equation is given as

$$\begin{Bmatrix} M_1 \\ M_2 \\ M_3 \\ M_4 \end{Bmatrix} = \left[\begin{bmatrix} r_{L1} & r_{L2} & r_{L3} & r_{L4} \\ r_{L1} & r_{L2} & r_{L3} & r_{L4} \\ r_{L1} & r_{L2} & r_{L3} & r_{L4} \\ r_{L1} & r_{L2} & r_{L3} & r_{L4} \end{bmatrix} - \begin{bmatrix} r_{M1} & r_{M1} & r_{M1} & r_{M1} \\ r_{M2} & r_{M2} & r_{M2} & r_{M2} \\ r_{M3} & r_{M3} & r_{M3} & r_{M3} \\ r_{M4} & r_{M4} & r_{M4} & r_{M4} \end{bmatrix} \right] \begin{Bmatrix} L_1 \\ L_2 \\ L_3 \\ L_4 \end{Bmatrix} \tag{A2}$$

In equation (A2) any elements of negative sign resulting from the subtraction are to be replaced by zeroes since the moment contribution of the loads inboard of a moment station is zero.

If the rectangular matrix of equation (A2) is symbolized by $[r]$, then equation (A2) may be written as

$$\left\{M\right\} = \left[r\right]\left\{L\right\} \tag{A3}$$

which is the matrix equivalent of

$$M_{rn} = \int_{r_n}^{R} l_V\left(r - r_n\right)dr$$

where the integral is to be evaluated at a selected number of stations.

The $[r]$ matrix is given in table V for the ten load and moment stations of figures 2 and 3. The matrix of table V may be extended to include more stations by expanding the equation $\left\{M\right\} = \left[r\right]\left\{L\right\}$ as shown in equation (A4), where the rectangular matrix postmultiplied by the diagonal matrix is equal to the $[r]$ matrix.

$$\begin{Bmatrix} M_{.85} \\ M_{.75} \\ M_{.65} \\ M_{.55} \\ M_{.45} \\ M_{.35} \\ M_{.25} \\ M_{.15} \\ M_{.0625} \\ M_{.0125} \\ \text{------} \\ M_0 \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 3 & 2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 4 & 3 & 2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 5 & 4 & 3 & 2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 6 & 5 & 4 & 3 & 2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 7 & 6 & 5 & 4 & 3 & 2 & 1 & 0 & 0 & 0 & 0 & 0 \\ 8 & 7 & 6 & 5 & 4 & 3 & 2 & 1 & 0 & 0 & 0 & 0 \\ 8.87 & 7.87 & 6.87 & 5.87 & 4.87 & 3.87 & 2.87 & 1.87 & 0.87 & 0 & 0 & 0 \\ 9.37 & 8.37 & 7.37 & 6.37 & 5.37 & 4.37 & 3.37 & 2.37 & 1.37 & 0 & 0.37 & 0 \\ \text{----} & \text{----} & \text{----} & \text{----} & \text{----} & \text{----} & \text{----} & \text{----} & \text{----} & \text{-} & \text{----} & \text{-} \\ 9.5 & 8.5 & 7.5 & 6.5 & 5.5 & 4.5 & 3.5 & 2.5 & 1.5 & 0 & 0.5 & 0 \end{bmatrix} \begin{bmatrix} \diagdown \\ \Delta r \\ \diagdown \end{bmatrix} \begin{Bmatrix} L_{.95} \\ L_{.85} \\ L_{.75} \\ L_{.65} \\ L_{.55} \\ L_{.45} \\ L_{.35} \\ L_{.25} \\ L_{.15} \\ L_{.0625} \\ L_{.05} \\ L_{.0125} \end{Bmatrix} \tag{A4}$$

To extend the matrix to more stations, continue the given number sequence of the rectangular matrix and obtain more rows at the top and more columns on the left. Keep unity numbers on the diagonal and zeroes on the right. The moment stations are then the given stations (shown in table V) multiplied by $10/n_e$, (where n_e is the new number of blade elements), and the obtained sequence is continued columnwise in order to provide additional stations. For example, the elements of the first column of a 20-station arrangement would be 1, 2, 3, 4, 5, 6, . . . 16, 17, 18.87, 19.37, and $19.5(r - h)/(R - h)$; the load stations would be 0.025, 0.075, 0.125, 0.175, . . . 0.925, and $0.975(r - h)/(R - h)$; and the moment stations would be 0, 0.00625, 0.03125, 0.0725, 0.125, 0.175, . . . 0.875, and $0.925(r - h)/(R - h)$.

Moment Due to Horizontal Forces Acting on a Deflected Blade

The general bending-moment equation of the text (eq. (1)) is

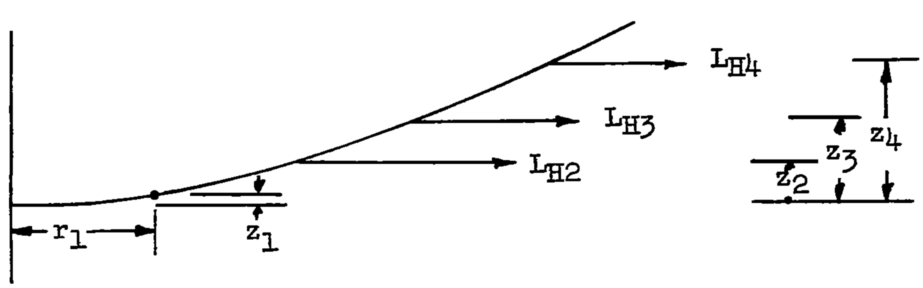
$$M_{rn} = \int_{r_n}^{R} l_V(r - r_n)dr - \int_{r_n}^{R} l_H(z - z_n)dr$$

In this section the matrix equivalent of the expression

$$M_{rn} = -\int_{r_n}^{R} l_H(z - z_n)dr \tag{A5}$$

is derived.

The system of equivalent horizontal concentrated loads, as shown in the following sketch, is again used, although some other representation might produce slightly more accurate results at the expense of simplicity.



At station r_1 the moment due to horizontal centrifugal forces may be written as

$$M_{r1} = -\left[L_{H4}\left(z_4 - z_1\right) + L_{H3}\left(z_3 - z_1\right) + L_{H2}\left(z_2 - z_1\right)\right] \tag{A6}$$

Collecting terms and writing equation (A6) in matrix form yields

$$\left\{M_{r1}\right\} = -\left\lfloor L_{H4} \ L_{H3} \ L_{H2} \ -\sum_{n=2}^{n=4} L_H \right\rceil \begin{Bmatrix} z_4 \\ z_3 \\ z_2 \\ z_1 \end{Bmatrix} \tag{A7}$$

If this procedure is now applied and extended to obtain the moments due to the centrifugal inertia forces at ten selected moment stations on the blade, the following matrix equation results:

$$\begin{Bmatrix} M_{1H10} \\ M_{1H9} \\ M_{1H8} \\ M_{1H7} \\ M_{1H6} \\ M_{1H5} \\ M_{1H4} \\ M_{1H3} \\ M_{1H2} \\ M_{1H1} \end{Bmatrix} = -\Omega^2 \begin{bmatrix}
F_{10} & -F_{10} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
F_{10} & F_9 & -\sum_{n=9}^{n=10} F_n & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
F_{10} & F_9 & F_8 & -\sum_{n=8}^{n=10} F_n & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
F_{10} & F_9 & F_8 & F_7 & -\sum_{n=7}^{n=10} F_n & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
F_{10} & F_9 & F_8 & F_7 & F_6 & -\sum_{n=6}^{n=10} F_n & 0 & 0 & 0 & 0 & 0 & 0 \\
F_{10} & F_9 & F_8 & F_7 & F_6 & F_5 & -\sum_{n=5}^{n=10} F_n & 0 & 0 & 0 & 0 & 0 \\
F_{10} & F_9 & F_8 & F_7 & F_6 & F_5 & F_4 & -\sum_{n=4}^{n=10} F_n & 0 & 0 & 0 & 0 \\
F_{10} & F_9 & F_8 & F_7 & F_6 & F_5 & F_4 & F_3 & -\sum_{n=3}^{n=10} F_n & 0 & 0 & 0 \\
F_{10} & F_9 & F_8 & F_7 & F_6 & F_5 & F_4 & F_3 & F_2 & -\sum_{n=2}^{n=10} F_n & 0 & 0 \\
F_{10} & F_9 & F_8 & F_7 & F_6 & F_5 & F_4 & F_3 & F_2 & 0 & F_1 & -\sum_{n=1}^{n=10} F_n
\end{bmatrix} \begin{Bmatrix} z_{.95} \\ z_{.85} \\ z_{.75} \\ z_{.65} \\ z_{.55} \\ z_{.45} \\ z_{.35} \\ z_{.25} \\ z_{.15} \\ z_{.0625} \\ z_{.05} \\ z_{.0125} \end{Bmatrix} \tag{A8}$$

where

$$F_{10} = m_{10}r_{10} = \frac{L_{H10}}{\Omega^2}$$

$$F_9 = m_9r_9 = \frac{L_{H9}}{\Omega^2}$$

$$\begin{matrix} \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{matrix}$$

$$F_1 = m_1r_1 = \frac{L_{H1}}{\Omega^2}$$

If the rectangular matrix of equation (A8) is symbolized by $\left[F\right]$ (given as table II), then equation (A8) may be written in shorter form as

$$\left\{M_{iH}\right\} = -\Omega^2\left[F\right]\left\{z\right\} \quad . \tag{A9}$$

which is the equation $M_{rn} = -\int_{r_n}^{R} l_H\left(z - z_n\right)dr$ in matrix form and is to be evaluated at selected stations. The moments M_{iH} are the moments on the blade caused by the centrifugal forces acting on the deflected blade.

In order to extend the matrix equation (A8) to n stations, make all of the negative summations on the shown diagonal go from F_n to F_k, where k is the subscript of the F values in the preceding column as illustrated. The elements off the diagonal of summations are obtained by extending the indicated sequence to F_n. The new moment and deflection stations are $10/n_e$ times the given stations, and the obtained sequence is extended columnwise for additional stations. The new blade-element arrangement has to have equal-length increments Δr.

Blade Deflection Due to Structural Moments

For the blade the general bending-moment equation of the text (eq. (7)) is given as

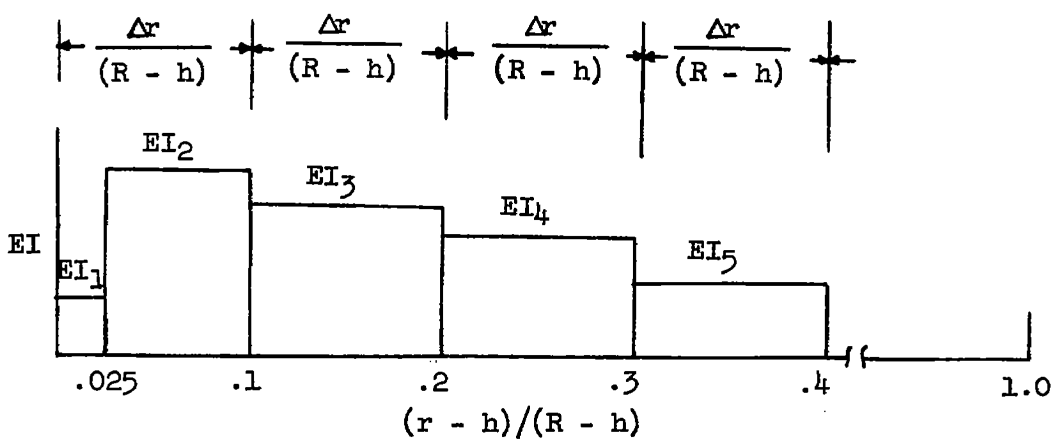
$$\left[1 + ig\right]\left\{M_s\right\} = [r]\left\{L_a\right\} + i\Omega\omega[r]\left[A\right]\left\{z\right\} + \omega^2[r]\left[m\right]\left\{z\right\} - \Omega^2[F]\left\{z\right\}$$

In subsequent derivations, a matrix expression for the deflection $\left\{z\right\}$ is required. The deflection $\left\{z\right\}$ is composed of a component due to the flexible-blade root slope and a component due to blade bending. In this section a matrix expression for the deflection component due to blade bending is derived. The derivation involves establishing the matrix equivalent of the analytical expression

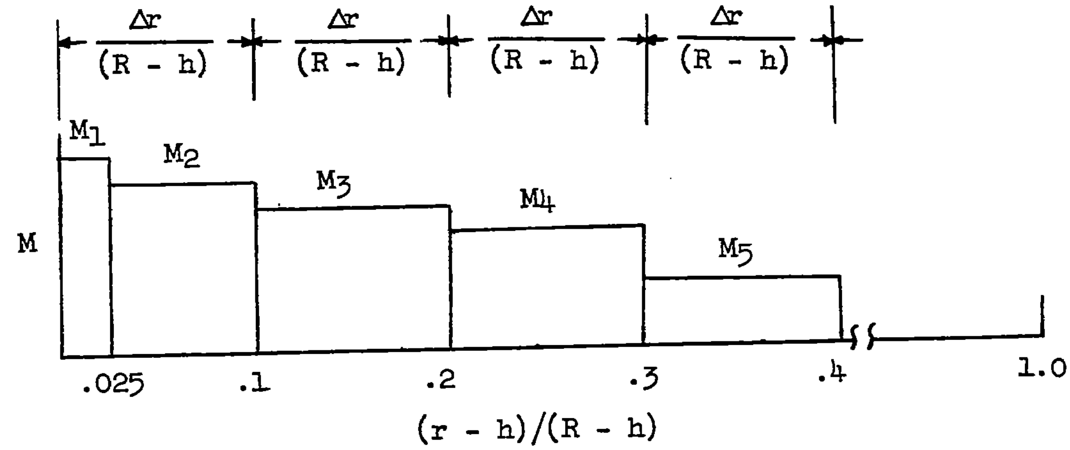
$$z_r = \int_0^r\int_0^r \frac{M}{EI}\,dr\,dr \tag{A10}$$

The derivation of the required matrix expression is illustrated herewith. The integration is first illustrated with four stations and then extended to more stations.

The blade stiffness and structural moment distributions are represented by the following step diagrams:



Blade stiffness distribution



Blade moment distribution

The segmental arrangement just noted could be made differently, but the particular arrangement shown was chosen for the reasons that: (1) it permits a large change of blade stiffness at the blade hub, and (2) it gives a moment station where there are large changes in moments when the methods are applied to the hinged blade.

The blade angles β_r resulting from the bending deflections are given by the equation $\beta_r = \int_0^r \frac{M}{EI}\,dr$ $\left(\text{or the eq. } \beta_r = \sum_0^r \frac{M}{EI}\,\Delta r\right)$ and can be deduced, for example, from the foregoing distributions as follows:

At $\frac{(r - h)}{(R - h)} = 0.025$,

$$\beta_r = \frac{M_1}{EI_1}\left(\frac{\Delta r}{4}\right)$$

At $\frac{(r - h)}{(R - h)} = 0.10$,

$$\beta_r = \frac{M_1}{EI_1}\left(\frac{\Delta r}{4}\right) + \frac{M_2}{EI_2}\left(\frac{3\ \Delta r}{4}\right)$$

At $\frac{(r - h)}{(R - h)} = 0.20$,

$$\beta_r = \frac{M_1}{EI_1}\left(\frac{\Delta r}{4}\right) + \frac{M_2}{EI_2}\left(\frac{3\ \Delta r}{4}\right) + \frac{M_3}{EI_3}(\Delta r)$$

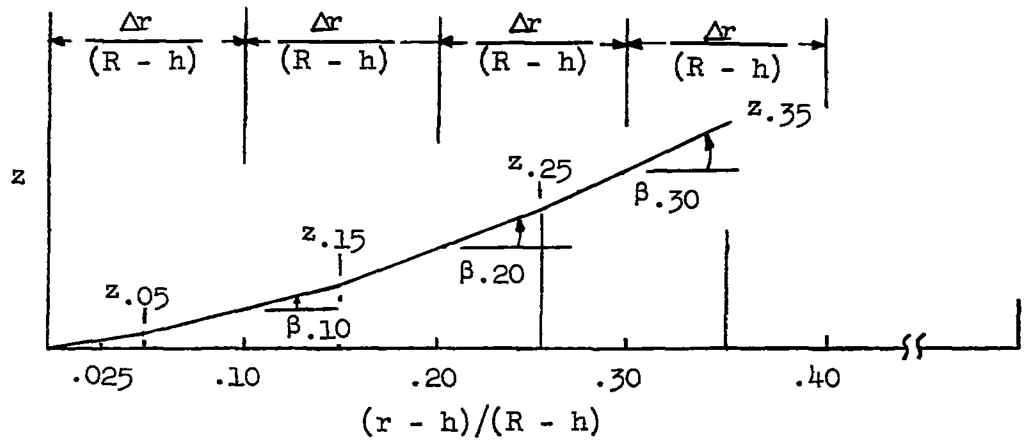
At $\frac{(r - h)}{(R - h)} = 0.30$,

$$\beta_r = \frac{M_1}{EI_1}\left(\frac{\Delta r}{4}\right) + \frac{M_2}{EI_2}\left(\frac{3\ \Delta r}{4}\right) + \frac{M_3}{EI_3}(\Delta r) + \frac{M_4}{EI_4}(\Delta r)$$

These distributions may be written in matrix form as follows:

$$\begin{Bmatrix} \beta_{.025} \\ \beta_{.10} \\ \beta_{.20} \\ \beta_{.30} \end{Bmatrix} = \begin{bmatrix} \frac{\Delta r}{4EI_1} & 0 & 0 & 0 \\ \frac{\Delta r}{4EI_1} & \frac{3\ \Delta r}{4EI_2} & 0 & 0 \\ \frac{\Delta r}{4EI_1} & \frac{3\ \Delta r}{4EI_2} & \frac{\Delta r}{EI_3} & 0 \\ \frac{\Delta r}{4EI_1} & \frac{3\ \Delta r}{4EI_2} & \frac{\Delta r}{EI_3} & \frac{\Delta r}{EI_4} \end{bmatrix} \begin{Bmatrix} M_1 \\ M_2 \\ M_3 \\ M_4 \end{Bmatrix} \tag{A11}$$

In order to extend the procedure to obtain the deflections in terms of the moments, the following sketch proves helpful:



At $\dfrac{(r - h)}{(R - h)} = 0.05$,

$$z = \beta_{0.025}\left(\frac{\Delta r}{2}\right) = \left(\frac{M_1}{EI_1}\,\frac{\Delta r}{4}\right)\frac{\Delta r}{2}$$

At $\dfrac{(r - h)}{(R - h)} = 0.15$,

$$z = z_{0.05} + \beta_{0.10}\,\Delta r = \left(\frac{M_1}{EI_1}\,\frac{\overline{\Delta r}^2}{8}\right) + \left(\frac{M_1\,\Delta r}{4EI_1} + \frac{3M_2\,\Delta r}{4EI_2}\right)\Delta r$$

At $\dfrac{(r - h)}{(R - h)} = 0.25$,

$$z = z_{0.15} + \beta_{0.20}\,\Delta r = \;.\;.\;.\;.\;.$$

Extending and collecting terms and writing in matrix form yields the following:

$$\begin{Bmatrix} z_{.05} \\ z_{.15} \\ z_{.25} \\ z_{.35} \end{Bmatrix} = \begin{bmatrix} \frac{\overline{\Delta r}^2}{8EI_1} & 0 & 0 & 0 \\ \frac{3\,\overline{\Delta r}^2}{8EI_1} & \frac{3\,\overline{\Delta r}^2}{4EI_2} & 0 & 0 \\ \frac{5\,\overline{\Delta r}^2}{8EI_1} & \frac{6\,\overline{\Delta r}^2}{4EI_2} & \frac{\overline{\Delta r}^2}{EI_3} & 0 \\ \frac{7\,\overline{\Delta r}^2}{8EI_1} & \frac{9\,\overline{\Delta r}^2}{4EI_2} & \frac{2\,\overline{\Delta r}^2}{EI_3} & \frac{\overline{\Delta r}^2}{EI_4} \end{bmatrix} \begin{Bmatrix} M_1 \\ M_2 \\ M_3 \\ M_4 \end{Bmatrix} \tag{A12}$$

Symbolizing the square matrix by $\left[Z_M\right]$ gives equation (A12) as

$$\left\{z\right\} = \left[Z_M\right]\left\{M_s\right\} \tag{A13}$$

Rearranging equation (A12) to have the blade tip values in the upper rows and extending to obtain deflection values at the twelve selected stations results in the following matrix. (See figs. 1, 2, and 3 for station location and the definition of the elements.)

$$
\begin{Bmatrix} z_{.95} \\ z_{.85} \\ z_{.75} \\ z_{.65} \\ z_{.55} \\ z_{.45} \\ z_{.35} \\ z_{.25} \\ z_{.15} \\ z_{.0625} \\ z_{.05} \\ z_{.0125} \end{Bmatrix}
=
\begin{bmatrix}
\frac{\overline{\Delta r}^2}{EI_{10}} & \frac{2\overline{\Delta r}^2}{EI_9} & \frac{3\overline{\Delta r}^2}{EI_8} & \frac{4\overline{\Delta r}^2}{EI_7} & \frac{5\overline{\Delta r}^2}{EI_6} & \frac{6\overline{\Delta r}^2}{EI_5} & \frac{7\overline{\Delta r}^2}{EI_4} & \frac{8\overline{\Delta r}^2}{EI_3} & \frac{27\overline{\Delta r}^2}{4EI_2} & \frac{19\overline{\Delta r}^2}{8EI_1} \\
0 & \frac{\overline{\Delta r}^2}{EI_9} & \frac{2\overline{\Delta r}^2}{EI_8} & \frac{3\overline{\Delta r}^2}{EI_7} & \frac{4\overline{\Delta r}^2}{EI_6} & \frac{5\overline{\Delta r}^2}{EI_5} & \frac{6\overline{\Delta r}^2}{EI_4} & \frac{7\overline{\Delta r}^2}{EI_3} & \frac{24\overline{\Delta r}^2}{4EI_2} & \frac{17\overline{\Delta r}^2}{8EI_1} \\
0 & 0 & \frac{\overline{\Delta r}^2}{EI_8} & \frac{2\overline{\Delta r}^2}{EI_7} & \frac{3\overline{\Delta r}^2}{EI_6} & \frac{4\overline{\Delta r}^2}{EI_5} & \frac{5\overline{\Delta r}^2}{EI_4} & \frac{6\overline{\Delta r}^2}{EI_3} & \frac{21\overline{\Delta r}^2}{4EI_2} & \frac{15\overline{\Delta r}^2}{8EI_1} \\
0 & 0 & 0 & \frac{\overline{\Delta r}^2}{EI_7} & \frac{2\overline{\Delta r}^2}{EI_6} & \frac{3\overline{\Delta r}^2}{EI_5} & \frac{4\overline{\Delta r}^2}{EI_4} & \frac{5\overline{\Delta r}^2}{EI_3} & \frac{18\overline{\Delta r}^2}{4EI_2} & \frac{13\overline{\Delta r}^2}{8EI_1} \\
0 & 0 & 0 & 0 & \frac{\overline{\Delta r}^2}{EI_6} & \frac{2\overline{\Delta r}^2}{EI_5} & \frac{3\overline{\Delta r}^2}{EI_4} & \frac{4\overline{\Delta r}^2}{EI_3} & \frac{15\overline{\Delta r}^2}{4EI_2} & \frac{11\overline{\Delta r}^2}{8EI_1} \\
0 & 0 & 0 & 0 & 0 & \frac{\overline{\Delta r}^2}{EI_5} & \frac{2\overline{\Delta r}^2}{EI_4} & \frac{3\overline{\Delta r}^2}{EI_3} & \frac{12\overline{\Delta r}^2}{4EI_2} & \frac{9\overline{\Delta r}^2}{8EI_1} \\
0 & 0 & 0 & 0 & 0 & 0 & \frac{\overline{\Delta r}^2}{EI_4} & \frac{2\overline{\Delta r}^2}{EI_3} & \frac{9\overline{\Delta r}^2}{4EI_2} & \frac{7\overline{\Delta r}^2}{8EI_1} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{\overline{\Delta r}^2}{EI_3} & \frac{6\overline{\Delta r}^2}{4EI_2} & \frac{5\overline{\Delta r}^2}{8EI_1} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{3\overline{\Delta r}^2}{4EI_2} & \frac{3\overline{\Delta r}^2}{8EI_1} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{3\overline{\Delta r}^2}{32EI_2} & \frac{5\overline{\Delta r}^2}{32EI_1} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{\overline{\Delta r}^2}{8EI_1} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{\overline{\Delta r}^2}{32EI_1}
\end{bmatrix}
\begin{Bmatrix} M_{s10} \\ M_{s9} \\ M_{s8} \\ M_{s7} \\ M_{s6} \\ M_{s5} \\ M_{s4} \\ M_{s3} \\ M_{s2} \\ M_{s1} \end{Bmatrix}
\qquad \text{(A14)}
$$

Equation (A14) may be written as follows:

$$
\begin{Bmatrix} z_{.95} \\ z_{.85} \\ z_{.75} \\ z_{.65} \\ z_{.55} \\ z_{.45} \\ z_{.35} \\ z_{.25} \\ z_{.15} \\ z_{.0625} \\ z_{.05} \\ z_{.0125} \end{Bmatrix}
=
\begin{bmatrix}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & \frac{27}{4} & \frac{19}{8} \\
0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & \frac{24}{4} & \frac{17}{8} \\
0 & 0 & 1 & 2 & 3 & 4 & 5 & 6 & \frac{21}{4} & \frac{15}{8} \\
0 & 0 & 0 & 1 & 2 & 3 & 4 & 5 & \frac{18}{4} & \frac{13}{8} \\
0 & 0 & 0 & 0 & 1 & 2 & 3 & 4 & \frac{15}{4} & \frac{11}{8} \\
0 & 0 & 0 & 0 & 0 & 1 & 2 & 3 & \frac{12}{4} & \frac{9}{8} \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 2 & \frac{9}{4} & \frac{7}{8} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & \frac{6}{4} & \frac{5}{8} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{3}{4} & \frac{3}{8} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{3}{32} & \frac{5}{32} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{8} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{32}
\end{bmatrix}
\left[\frac{\overline{\Delta r}^2}{EI} \right]
\begin{Bmatrix} M_{s10} \\ M_{s9} \\ M_{s8} \\ M_{s7} \\ M_{s6} \\ M_{s5} \\ M_{s4} \\ M_{s3} \\ M_{s2} \\ M_{s1} \end{Bmatrix}
\tag{A15}
$$

To extend to more stations, simply extend the established number sequence in the rectangular matrix. Keep unit values on the extended diagonal and zeroes below. The procedure for obtaining the new moment and deflection stations is as previously discussed in the $\lceil r \rfloor$ and $\lceil F \rfloor$ matrix derivations. The values of EI must pertain to the new moment stations.

The results obtained by using either equation (A14) or (A15) have been compared with results obtained by graphical numerical double integration of a known M and EI distribution, and the results agree very well. If desired, the elements of equation (A14) or (A15) could be determined experimentally from the actual blade.

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TABLE I.- MODIFIED IDENTITY MATRIX $\begin{bmatrix} [1.0] & \{0\} \\ \lfloor 0 \rfloor & 0 \end{bmatrix}$

Moment station, (r - h)/(R - h)											Moment station, $\frac{r - h}{R - h}$
0.85	0.75	0.65	0.55	0.45	0.35	0.25	0.15	0.0625	0.0125	β_h	
1.0	0	0	0	0	0	0	0	0	0	0	0.85
0	1.0	0	0	0	0	0	0	0	0	0	.75
0	0	1.0	0	0	0	0	0	0	0	0	.65
0	0	0	1.0	0	0	0	0	0	0	0	.55
0	0	0	0	1.0	0	0	0	0	0	0	.45
0	0	0	0	0	1.0	0	0	0	0	0	.35
0	0	0	0	0	0	1.0	0	0	0	0	.25
0	0	0	0	0	0	0	1.0	0	0	0	.15
0	0	0	0	0	0	0	0	1.0	0	0	.0625
0	0	0	0	0	0	0	0	0	1.0	0	.0125
0	0	0	0	0	0	0	0	0	0	0	0

TABLE II.- CENTRIFUGAL-FORCE MATRIX [F]

$[F_n = m_n r_n]$

Deflection station, (r - h)/(R - h)												Moment station, $\frac{r - h}{R - h}$
0.95	0.85	0.75	0.65	0.55	0.45	0.35	0.25	0.15	0.0625	0.05	0.0125	
F_{10}	$-F_{10}$	0	0	0	0	0	0	0	0	0	0	0.85
F_{10}	F_9	$-\sum_{n=9}^{n=10} F_n$	0	0	0	0	0	0	0	0	0	.75
F_{10}	F_9	F_8	$-\sum_{n=8}^{n=10} F_n$	0	0	0	0	0	0	0	0	.65
F_{10}	F_9	F_8	F_7	$-\sum_{n=7}^{n=10} F_n$	0	0	0	0	0	0	0	.55
F_{10}	F_9	F_8	F_7	F_6	$-\sum_{n=6}^{n=10} F_n$	0	0	0	0	0	0	.45
F_{10}	F_9	F_8	F_7	F_6	F_5	$-\sum_{n=5}^{n=10} F_n$	0	0	0	0	0	.35
F_{10}	F_9	F_8	F_7	F_6	F_5	F_4	$-\sum_{n=4}^{n=10} F_n$	0	0	0	0	.25
F_{10}	F_9	F_8	F_7	F_6	F_5	F_4	F_3	$-\sum_{n=3}^{n=10} F_n$	0	0	0	.15
F_{10}	F_9	F_8	F_7	F_6	F_5	F_4	F_3	F_2	$-\sum_{n=2}^{n=10} F_n$	0	0	.0625
F_{10}	F_9	F_8	F_7	F_6	F_5	F_4	F_3	F_2	0	F_1	$-\sum_{n=1}^{n=10} F_n$	.0125
F_{10}	F_9	F_8	F_7	F_6	F_5	F_4	F_3	F_2	0	F_1	0	0

TABLE III.- DEFLECTION MATRIX $\left[Z_M \,\vdots\, \{r - h\}\right]$

Moment station, $(r - h)/(R - h)$										δ_h	Deflection station, $\frac{r - h}{R - h}$
0.85	0.75	0.65	0.55	0.45	0.35	0.25	0.15	0.0625	0.0125		
$\frac{\Delta r^2}{EI_{10}}$	$\frac{2\Delta r^2}{EI_9}$	$\frac{3\Delta r^2}{EI_8}$	$\frac{4\Delta r^2}{EI_7}$	$\frac{5\Delta r^2}{EI_6}$	$\frac{6\Delta r^2}{EI_5}$	$\frac{7\Delta r^2}{EI_4}$	$\frac{8\Delta r^2}{EI_3}$	$\frac{27\Delta r^2}{4EI_2}$	$\frac{19\Delta r^2}{8EI_1}$	$(r - h)_{0.95}$	0.95
0	$\frac{\Delta r^2}{EI_9}$	$\frac{2\Delta r^2}{EI_8}$	$\frac{3\Delta r^2}{EI_7}$	$\frac{4\Delta r^2}{EI_6}$	$\frac{5\Delta r^2}{EI_5}$	$\frac{6\Delta r^2}{EI_4}$	$\frac{7\Delta r^2}{EI_3}$	$\frac{24\Delta r^2}{4EI_2}$	$\frac{17\Delta r^2}{8EI_1}$	$(r - h)_{0.85}$	.85
0	0	$\frac{\Delta r^2}{EI_8}$	$\frac{2\Delta r^2}{EI_7}$	$\frac{3\Delta r^2}{EI_6}$	$\frac{4\Delta r^2}{EI_5}$	$\frac{5\Delta r^2}{EI_4}$	$\frac{6\Delta r^2}{EI_3}$	$\frac{21\Delta r^2}{4EI_2}$	$\frac{15\Delta r^2}{8EI_1}$	$(r - h)_{0.75}$	.75
0	0	0	$\frac{\Delta r^2}{EI_7}$	$\frac{2\Delta r^2}{EI_6}$	$\frac{3\Delta r^2}{EI_5}$	$\frac{4\Delta r^2}{EI_4}$	$\frac{5\Delta r^2}{EI_3}$	$\frac{18\Delta r^2}{4EI_2}$	$\frac{13\Delta r^2}{8EI_1}$	$(r - h)_{0.65}$	.65
0	0	0	0	$\frac{\Delta r^2}{EI_6}$	$\frac{2\Delta r^2}{EI_5}$	$\frac{3\Delta r^2}{EI_4}$	$\frac{4\Delta r^2}{EI_3}$	$\frac{15\Delta r^2}{4EI_2}$	$\frac{11\Delta r^2}{8EI_1}$	$(r - h)_{0.55}$	.55
0	0	0	0	0	$\frac{\Delta r^2}{EI_5}$	$\frac{2\Delta r^2}{EI_4}$	$\frac{3\Delta r^2}{EI_3}$	$\frac{12\Delta r^2}{4EI_2}$	$\frac{9\Delta r^2}{8EI_1}$	$(r - h)_{0.45}$	.45
0	0	0	0	0	0	$\frac{\Delta r^2}{EI_4}$	$\frac{2\Delta r^2}{EI_3}$	$\frac{9\Delta r^2}{4EI_2}$	$\frac{7\Delta r^2}{8EI_1}$	$(r - h)_{0.35}$	.35
0	0	0	0	0	0	0	$\frac{\Delta r^2}{EI_3}$	$\frac{6\Delta r^2}{4EI_2}$	$\frac{5\Delta r^2}{8EI_1}$	$(r - h)_{0.25}$	.25
0	0	0	0	0	0	0	0	$\frac{3\Delta r^2}{4EI_2}$	$\frac{3\Delta r^2}{8EI_1}$	$(r - h)_{0.15}$	.15
0	0	0	0	0	0	0	0	$\frac{3\Delta r^2}{32EI_2}$	$\frac{5\Delta r^2}{32EI_1}$	$(r - h)_{0.0625}$	.0625
0	0	0	0	0	0	0	0	0	$\frac{\Delta r^2}{8EI_1}$	$(r - h)_{0.05}$	.05
0	0	0	0	0	0	0	0	0	$\frac{\Delta r^2}{32EI_1}$	$(r - h)_{0.0125}$	.0125

TABLE IV.- VERTICAL INERTIA MATRIX $\left[M_{zV}\right]$

Deflection station, $(r - h)/(R - h)$												Moment station, $\frac{r - h}{R - h}$
0.95	0.85	0.75	0.65	0.55	0.45	0.35	0.25	0.15	0.0625	0.05	0.0125	
$\overline{\Delta rm}_{10}$	0	0	0	0	0	0	0	0	0	0	0	0.85
$2\overline{\Delta rm}_{10}$	$\overline{\Delta rm}_{9}$	0	0	0	0	0	0	0	0	0	0	.75
$3\overline{\Delta rm}_{10}$	$2\overline{\Delta rm}_{9}$	$\overline{\Delta rm}_{8}$	0	0	0	0	0	0	0	0	0	.65
$4\overline{\Delta rm}_{10}$	$3\overline{\Delta rm}_{9}$	$2\overline{\Delta rm}_{8}$	$\overline{\Delta rm}_{7}$	0	0	0	0	0	0	0	0	.55
$5\overline{\Delta rm}_{10}$	$4\overline{\Delta rm}_{9}$	$3\overline{\Delta rm}_{8}$	$2\overline{\Delta rm}_{7}$	$\overline{\Delta rm}_{6}$	0	0	0	0	0	0	0	.45
$6\overline{\Delta rm}_{10}$	$5\overline{\Delta rm}_{9}$	$4\overline{\Delta rm}_{8}$	$3\overline{\Delta rm}_{7}$	$2\overline{\Delta rm}_{6}$	$\overline{\Delta rm}_{5}$	0	0	0	0	0	0	.35
$7\overline{\Delta rm}_{10}$	$6\overline{\Delta rm}_{9}$	$5\overline{\Delta rm}_{8}$	$4\overline{\Delta rm}_{7}$	$3\overline{\Delta rm}_{6}$	$2\overline{\Delta rm}_{5}$	$\overline{\Delta rm}_{4}$	0	0	0	0	0	.25
$8\overline{\Delta rm}_{10}$	$7\overline{\Delta rm}_{9}$	$6\overline{\Delta rm}_{8}$	$5\overline{\Delta rm}_{7}$	$4\overline{\Delta rm}_{6}$	$3\overline{\Delta rm}_{5}$	$2\overline{\Delta rm}_{4}$	$\overline{\Delta rm}_{3}$	0	0	0	0	.15
$8.87\overline{\Delta rm}_{10}$	$7.87\overline{\Delta rm}_{9}$	$6.87\overline{\Delta rm}_{8}$	$5.87\overline{\Delta rm}_{7}$	$4.87\overline{\Delta rm}_{6}$	$3.87\overline{\Delta rm}_{5}$	$2.87\overline{\Delta rm}_{4}$	$1.87\overline{\Delta rm}_{3}$	$.87\overline{\Delta rm}_{2}$	0	0	0	.0625
$9.37\overline{\Delta rm}_{10}$	$8.37\overline{\Delta rm}_{9}$	$7.37\overline{\Delta rm}_{8}$	$6.37\overline{\Delta rm}_{7}$	$5.37\overline{\Delta rm}_{6}$	$4.37\overline{\Delta rm}_{5}$	$3.37\overline{\Delta rm}_{4}$	$2.37\overline{\Delta rm}_{3}$	$1.37\overline{\Delta rm}_{2}$	0	$.37\overline{\Delta rm}_{1}$	0	.0125
$9.5\overline{\Delta rm}_{10}$	$8.5\overline{\Delta rm}_{9}$	$7.5\overline{\Delta rm}_{8}$	$6.5\overline{\Delta rm}_{7}$	$5.5\overline{\Delta rm}_{6}$	$4.5\overline{\Delta rm}_{5}$	$3.5\overline{\Delta rm}_{4}$	$2.5\overline{\Delta rm}_{3}$	$1.5\overline{\Delta rm}_{2}$	0	$.5\overline{\Delta rm}_{1}$	0	0

TABLE V.- INTEGRATING MATRIX [r]

Load station, (r - h)/(R - h)												Moment station, $\frac{r - h}{R - h}$
0.95	0.85	0.75	0.65	0.55	0.45	0.35	0.25	0.15	0.0625	0.05	0.0125	
Δr	0	0	0	0	0	0	0	0	0	0	0	0.85
2 Δr	Δr	0	0	0	0	0	0	0	0	0	0	.75
3 Δr	2 Δr	Δr	0	0	0	0	0	0	0	0	0	.65
4 Δr	3 Δr	2 Δr	Δr	0	0	0	0	0	0	0	0	.55
5 Δr	4 Δr	3 Δr	2 Δr	Δr	0	0	0	0	0	0	0	.45
6 Δr	5 Δr	4 Δr	3 Δr	2 Δr	Δr	0	0	0	0	0	0	.35
7 Δr	6 Δr	5 Δr	4 Δr	3 Δr	2 Δr	Δr	0	0	0	0	0	.25
8 Δr	7 Δr	6 Δr	5 Δr	4 Δr	3 Δr	2 Δr	Δr	0	0	0	0	.15
8.87 Δr	7.87 Δr	6.87 Δr	5.87 Δr	4.87 Δr	3.87 Δr	2.87 Δr	1.87 Δr	.87 Δr	0	0	0	.0625
9.37 Δr	8.37 Δr	7.37 Δr	6.37 Δr	5.37 Δr	4.37 Δr	3.37 Δr	2.37 Δr	1.37 Δr	0	.37 Δr	0	.0125
9.5 Δr	8.5 Δr	7.5 Δr	6.5 Δr	5.5 Δr	4.5 Δr	3.5 Δr	2.5 Δr	1.5 Δr	0	.5 Δr	0	0

TABLE VI.- AERODYNAMIC DAMPING MATRIX [A]

$$\left[K = -\frac{1}{2}\rho a\,\Delta r\right]$$

Deflection station, (r - h)/(R - h)												Load station, $\frac{r-h}{R-h}$
0.95	0.85	0.75	0.65	0.55	0.45	0.35	0.25	0.15	0.0625	0.05	0.0125	
$Kc_{10}r_{10}$	0	0	0	0	0	0	0	0	0	0	0	0.95
0	Kc_9r_9	0	0	0	0	0	0	0	0	0	0	.85
0	0	Kc_8r_8	0	0	0	0	0	0	0	0	0	.75
0	0	0	Kc_7r_7	0	0	0	0	0	0	0	0	.65
0	0	0	0	Kc_6r_6	0	0	0	0	0	0	0	.55
0	0	0	0	0	Kc_5r_5	0	0	0	0	0	0	.45
0	0	0	0	0	0	Kc_4r_4	0	0	0	0	0	.35
0	0	0	0	0	0	0	Kc_3r_3	0	0	0	0	.25
0	0	0	0	0	0	0	0	Kc_2r_2	0	0	0	.15
0	0	0	0	0	0	0	0	0	0	0	0	.0625
0	0	0	0	0	0	0	0	0	0	Kc_1r_1	0	.05
0	0	0	0	0	0	0	0	0	0	0	0	.0125

TABLE VII.- CENTRIFUGAL-FORCE MATRIX PARAMETER $\Omega^2[F]$, lb, FOR EXAMPLE BLADE

$\left[\Omega^2 = 4257.027 \text{ (radians/sec)}^2\right]$

Deflection station, (r - h)/(R - h)												Moment station, $\frac{r - h}{R - h}$
0.95	0.85	0.75	0.65	0.55	0.45	0.35	0.25	0.15	0.0625	0.05	0.0125	
2144.988	-2144.988	0	0	0	0	0	0	0	0	0	0	0.85
2144.988	1927.752	-4072.740	0	0	0	0	0	0	0	0	0	.75
2144.988	1927.752	1700.937	-5773.677	0	0	0	0	0	0	0	0	.65
2144.988	1927.752	1700.937	1113.766	-6887.445	0	0	0	0	0	0	0	.55
2144.988	1927.752	1700.937	1113.766	1253.056	-8140.499	0	0	0	0	0	0	.45
2144.988	1927.752	1700.937	1113.766	1253.056	789.338	-8929.857	0	0	0	0	0	.35
2144.988	1927.752	1700.937	1113.766	1253.056	789.338	832.589	-9762.426	0	0	0	0	.25
2144.988	1927.752	1700.937	1113.766	1253.056	789.338	832.589	647.621	-10410.047	0	0	0	.15
2144.988	1927.752	1700.937	1113.766	1253.056	789.338	832.589	647.621	1995.864	-12405.911	0	0	.0625
2144.988	1927.752	1700.937	1113.766	1253.056	789.338	832.589	647.621	1995.864	0	1533.551	-13939.462	.0125
2144.988	1927.752	1700.937	1113.766	1253.056	789.338	832.589	647.621	1995.864	0	1533.551	0	0

TABLE VIII.- VERTICAL-INERTIA MATRIX $\left[M_{zV}\right]$, lb-sec^2, FOR EXAMPLE BLADE

Deflection station, (r - h)/(R - h)												Moment station, $\frac{r-h}{R-h}$
0.95	0.85	0.75	0.65	0.55	0.45	0.35	0.25	0.15	0.0625	0.05	0.0125	
0.053038	0	0	0	0	0	0	0	0	0	0	0	0.85
.106077	.053274	0	0	0	0	0	0	0	0	0	0	.75
.159116	.106549	.053274	0	0	0	0	0	0	0	0	0	.65
.212155	.159824	.106549	.040251	0	0	0	0	0	0	0	0	.55
.265194	.213099	.159824	.080503	.053519	0	0	0	0	0	0	0	.45
.318233	.266375	.213099	.120755	.107038	.041205	0	0	0	0	0	0	.35
.371272	.319649	.266375	.161007	.160557	.082410	.055882	0	0	0	0	0	.25
.424311	.372924	.319649	.201258	.214077	.123615	.111765	.060854	0	0	0	0	.15
.470718	.419538	.366263	.236478	.260869	.159669	.160661	.114101	.273497	0	0	0	.0625
.497238	.446176	.392901	.256603	.287664	.180271	.188602	.144528	.429787	0	.270199	0	.0125
.503869	.452837	.399562	.261636	.294355	.185423	.195589	.152137	.468867	0	.360292	0	0

TABLE IX.- DEFLECTION MATRIX $\left[\left[Z_M \times 10^6\right] \vdots \left\{r - h\right\}\right]$ FOR EXAMPLE BLADE $\left[(\text{in./lb-in.}) \times 10^6 \vdots \text{in.}\right]$

Moment station, (r - h)/(R - h)										β_h	Deflection station, $\frac{r - h}{R - h}$
0.85	0.75	0.65	0.55	0.45	0.35	0.25	0.15	0.0625	0.0125		
26.163	52.326	78.489	104.652	116.245	93.030	93.009	38.712	12.825	7.619	86.925	0.95
0	26.163	52.326	78.489	92.996	77.525	79.722	33.873	11.400	6.817	77.775	.85
0	0	26.163	52.326	69.747	62.020	66.435	29.034	9.975	6.015	68.625	.75
0	0	0	26.163	46.498	46.515	53.148	24.195	8.550	5.213	59.475	.65
0	0	0	0	23.249	31.010	39.861	19.356	7.125	4.411	50.325	.55
0	0	0	0	0	15.505	26.574	14.517	5.700	3.609	41.175	.45
0	0	0	0	0	0	13.287	9.678	4.275	2.807	32.025	.35
0	0	0	0	0	0	0	4.839	2.850	2.005	22.875	.25
0	0	0	0	0	0	0	0	1.425	1.203	13.725	.15
0	0	0	0	0	0	0	0	.178	.501	5.718	.0625
0	0	0	0	0	0	0	0	0	.401	4.575	.05
0	0	0	0	0	0	0	0	0	.100	1.144	.0125

TABLE X.- INTEGRATING MATRIX [r], in., FOR EXAMPLE BLADE

Load station, $(r - h)/(R - h)$												Moment station, $\frac{r - h}{R - h}$
0.95	0.85	0.75	0.65	0.55	0.45	0.35	0.25	0.15	0.0625	0.05	0.0125	
9.150	0	0	0	0	0	0	0	0	0	0	0	0.85
18.300	9.150	0	0	0	0	0	0	0	0	0	0	.75
27.450	18.300	9.150	0	0	0	0	0	0	0	0	0	.65
36.600	27.450	18.300	9.150	0	0	0	0	0	0	0	0	.55
45.750	36.600	27.450	18.300	9.150	0	0	0	0	0	0	0	.45
54.900	45.750	36.600	27.450	18.300	9.150	0	0	0	0	0	0	.35
64.050	54.900	45.750	36.600	27.450	18.300	9.150	0	0	0	0	0	.25
73.200	64.050	54.900	45.750	36.600	27.450	18.300	9.150	0	0	0	0	.15
81.206	72.056	62.906	53.756	44.600	35.456	26.306	17.156	8.006	0	0	0	.0625
85.781	76.631	67.481	58.331	49.181	40.031	30.881	21.731	12.581	0	3.431	0	.0125
86.925	77.775	68.625	59.475	50.325	41.175	32.025	22.875	13.725	0	4.575	0	0

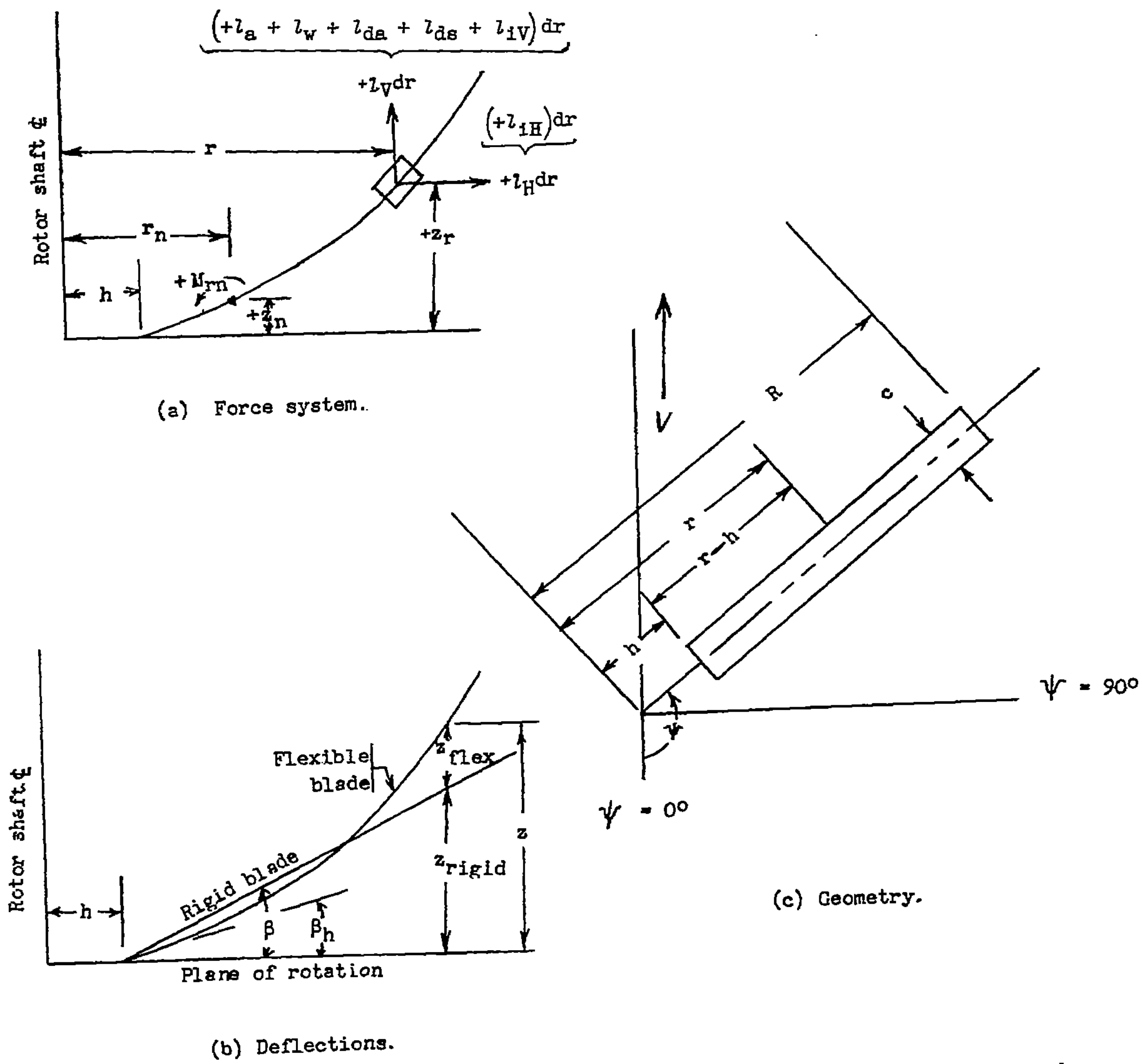


Figure 1.- Definitions of force and geometric symbols.

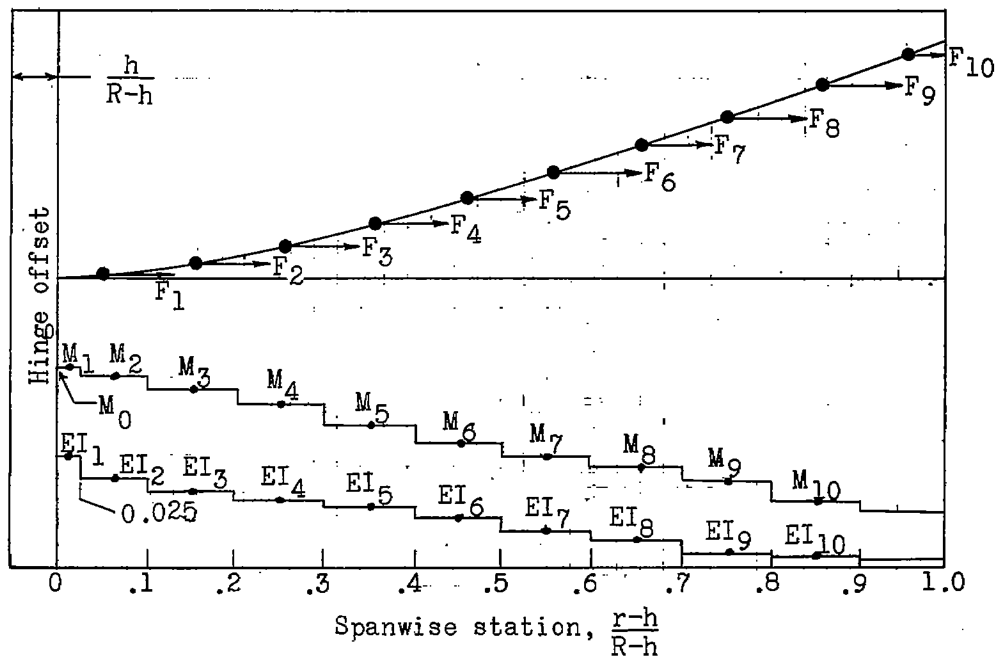


Figure 2.- Hinged-blade segmental arrangement for the matrix tables.

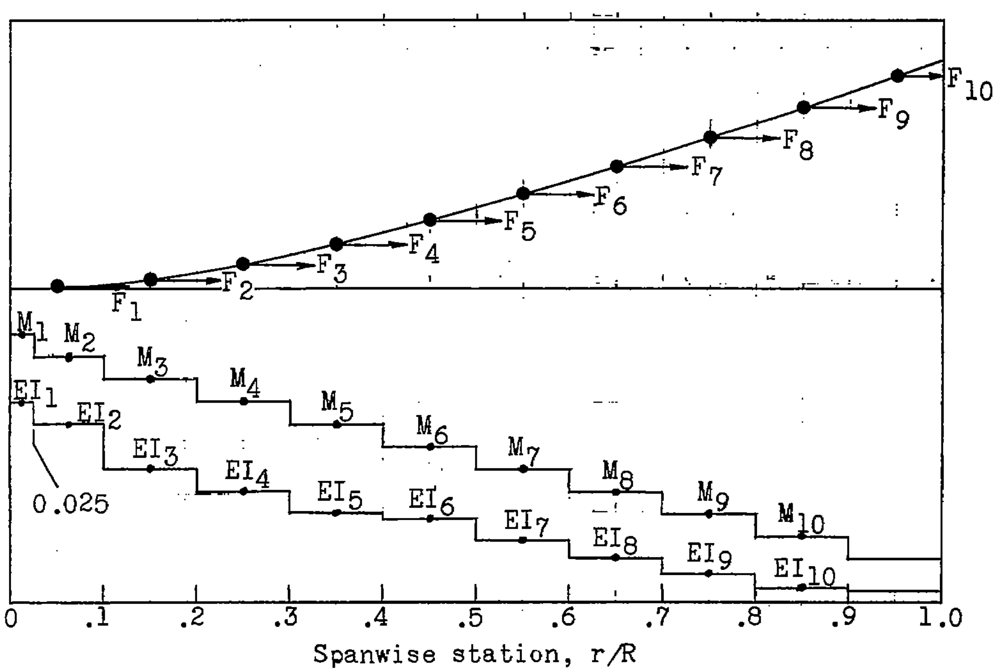


Figure 3.- Cantilever-blade segmental arrangement for the matrix tables.

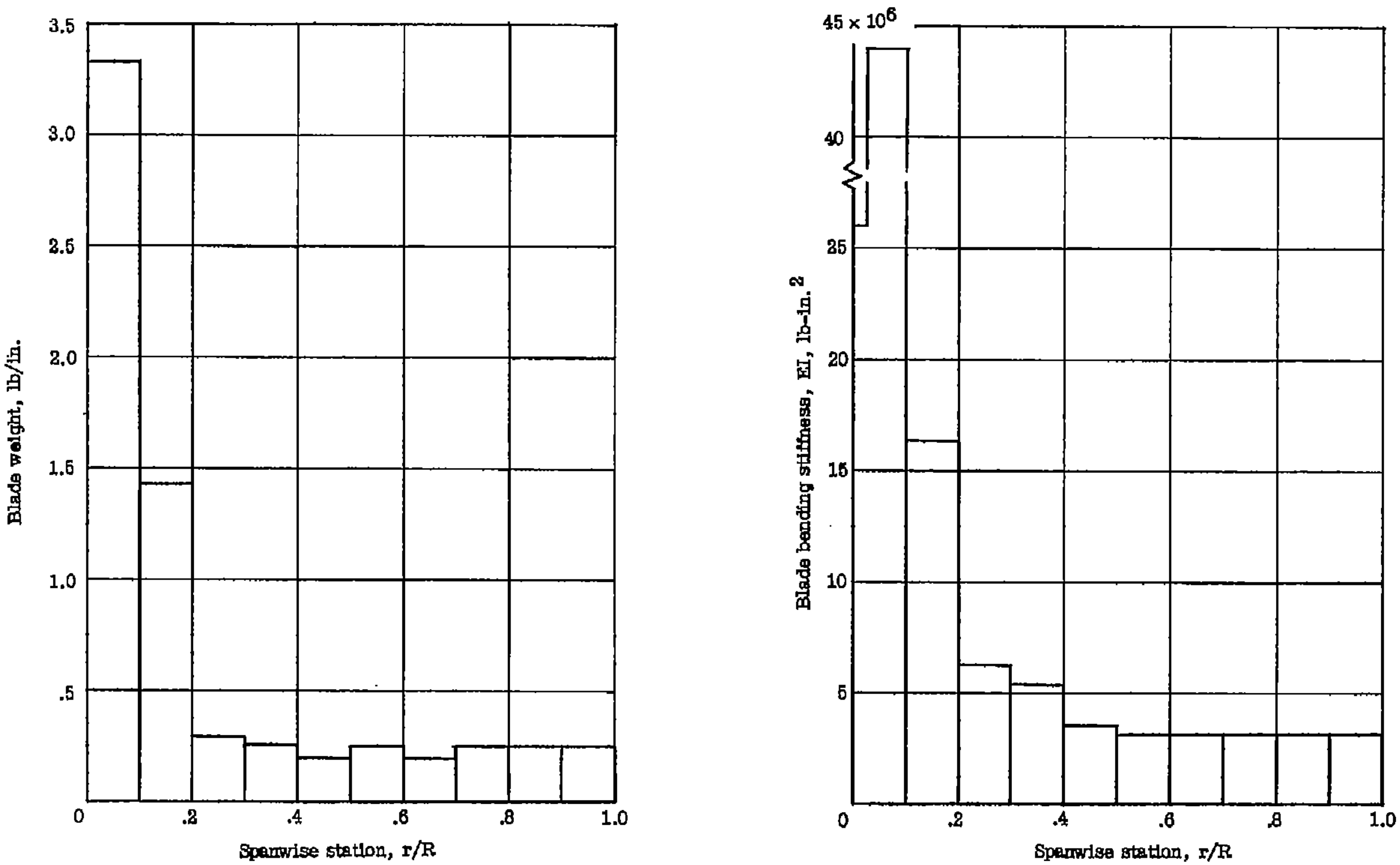
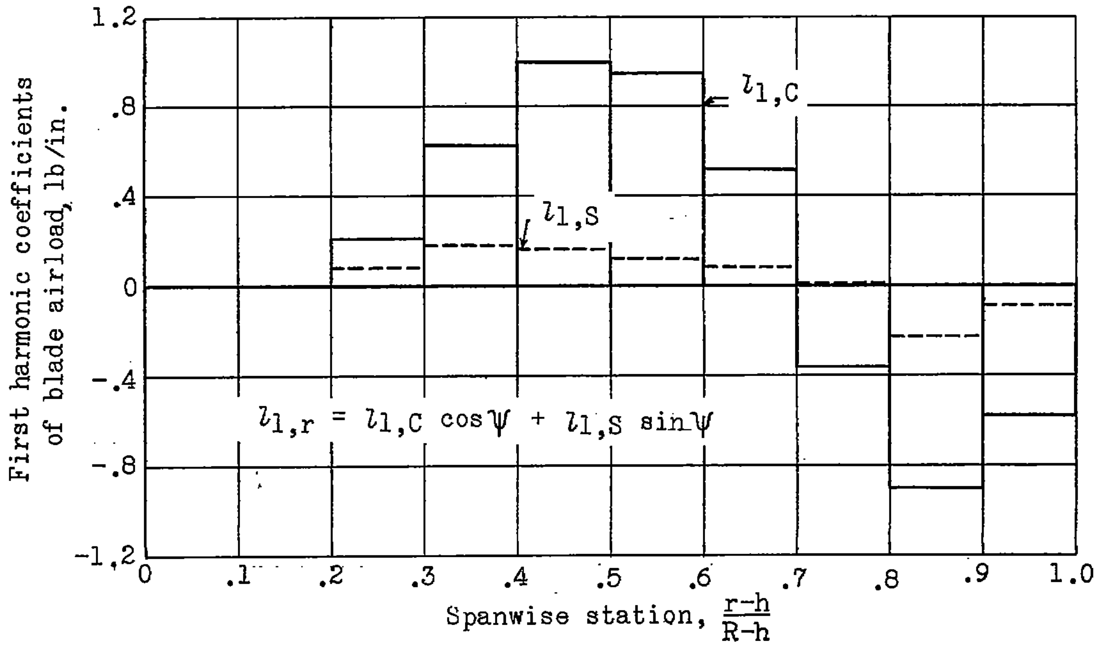
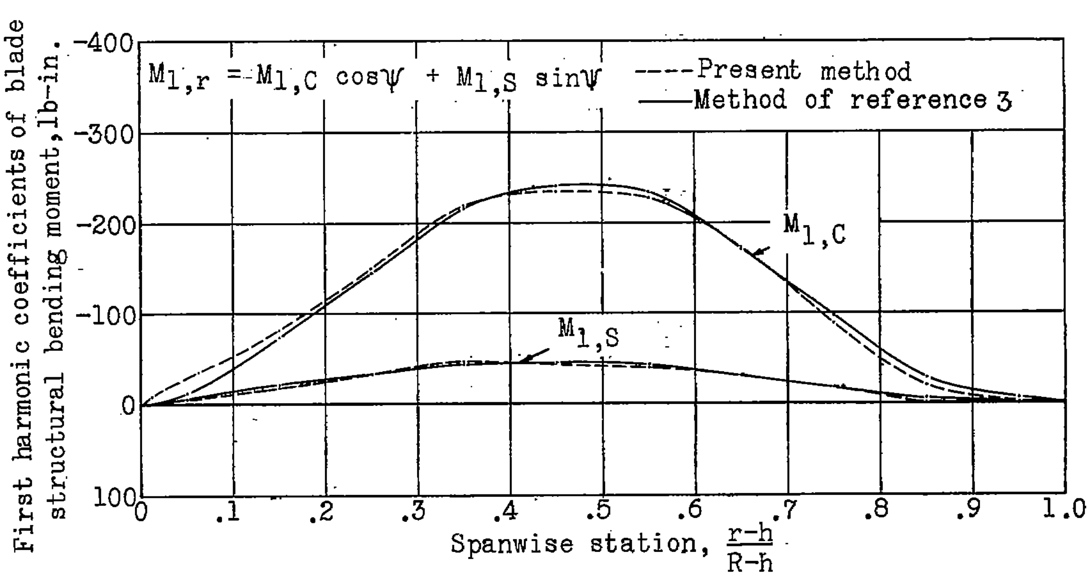


Figure 4.- Blade weight and stiffness used in the calculations.

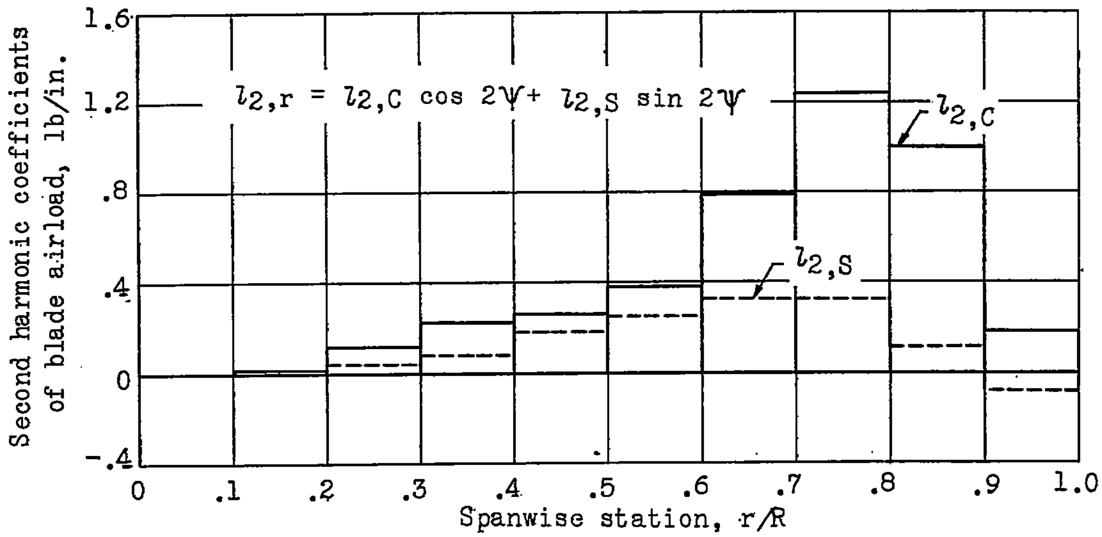


(a) Spanwise variation of first harmonic coefficients of blade airload.

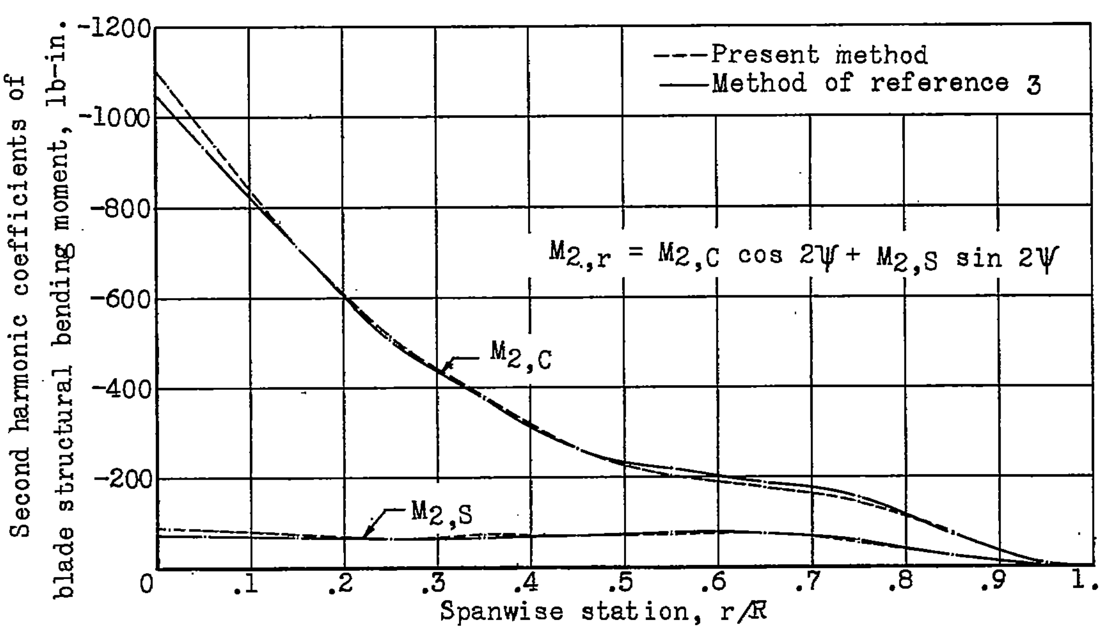


(b) Calculated spanwise variation of first harmonic structural moment coefficients due to first harmonic airload coefficients.

Figure 5.- Airloads and structural moments for hinged blade used in numerical examples.



(a) Spanwise variation of second harmonic coefficients of blade airload.



(b) Calculated spanwise variation of second harmonic structural moment coefficients due to second harmonic airload coefficients.

Figure 6.- Airloads and structural moments for cantilever blade used in numerical examples.

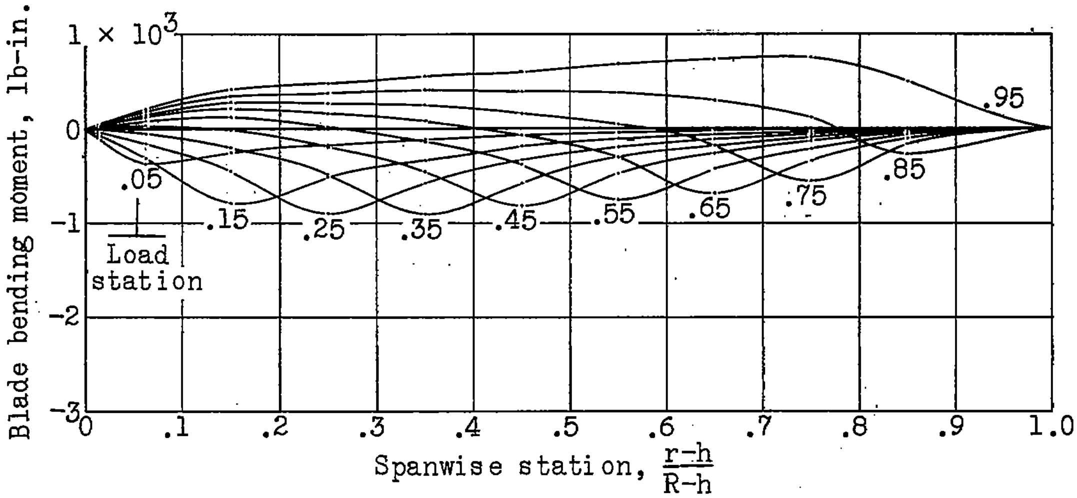


Figure 7.- Structural moment on zero-offset rotating hinged blade due to 100-pound concentrated loads at various stations.

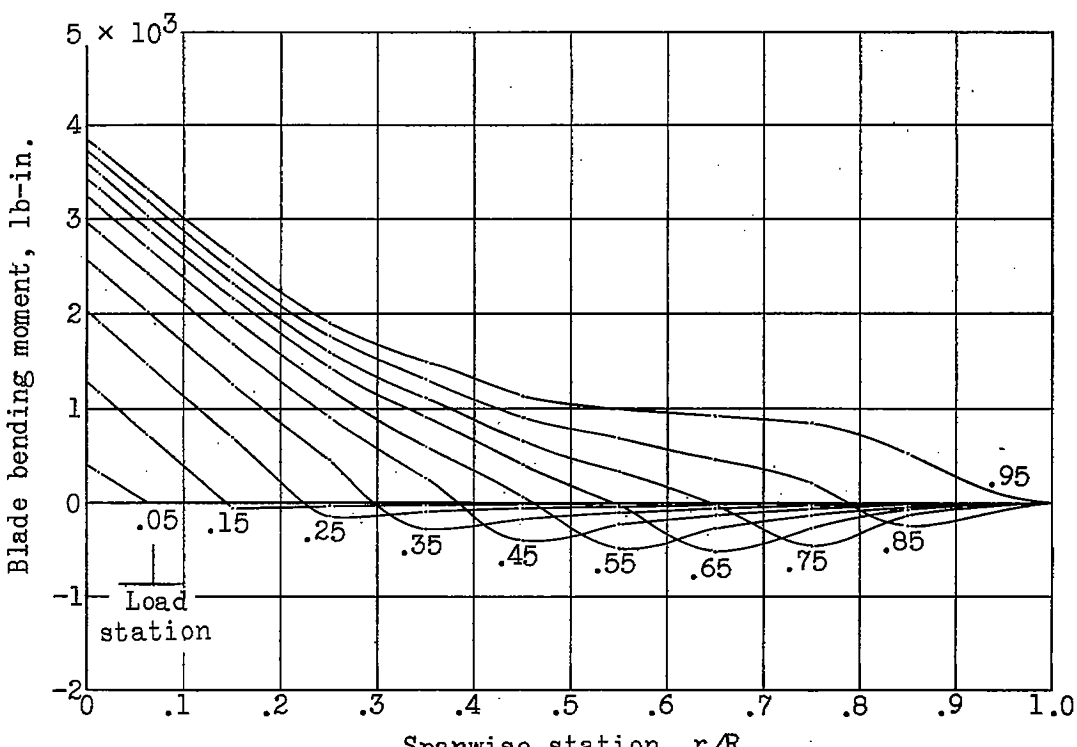


Figure 8.- Structural moment on rotating cantilever blade due to 100-pound concentrated loads at various stations.

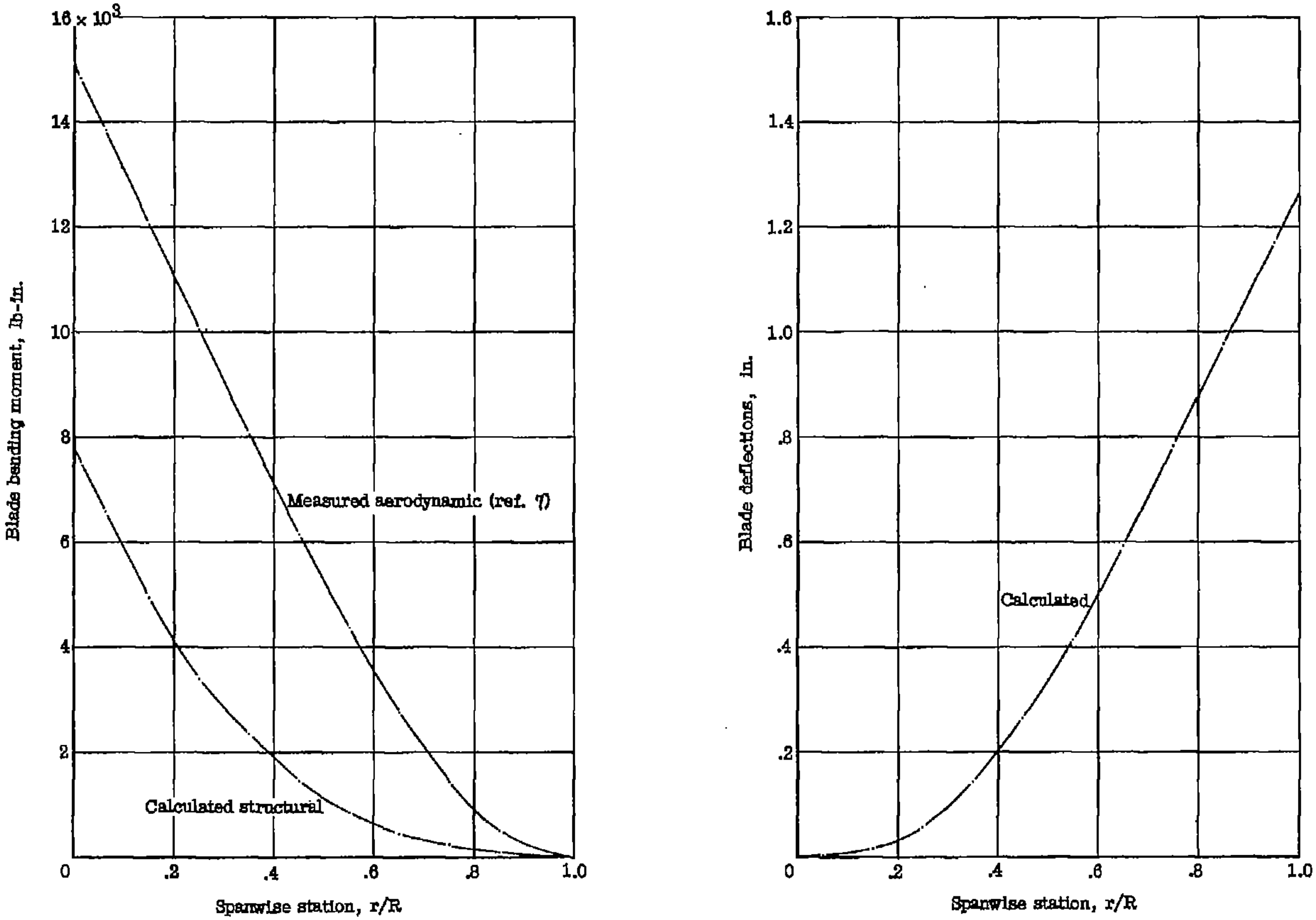


Figure 9.- Bending moment and deflections on cantilever blade in hovering. Thrust = 490 lb; ΩR = 497 ft/sec.

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